CR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KÄHLER PRODUCT MANIFOLDS

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ABSTRACT

In this article we investigate the geometry of CR-lightlike submanifolds in an indefinite Kähler product manifold. In particular, we obtain the necessary and sufficient conditions for a CR-lightlike submanifold in an indefinite Kähler product manifold to be either CR-lightlike product, or $D$-geodesic, or $D'$-geodesic. We also study totally umbilical and curvature-invariant CR-lightlike submanifolds in $\bar{M}(c_1) \times \bar{M}(c_2)$.

Key words: indefinite Kähler manifold, CR-lightlike submanifold, CR-lightlike product, real space form, and complex space form

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1. INTRODUCTION

The general theory of lightlike submanifolds has been developed mainly by Bejancu and Duggal [7, 8]. They constructed principal vector bundles to a lightlike submanifold in a semi-Riemannian manifold and obtained Gauss–Weingarten formulae as well as other properties of this submanifold [7].

The study of the geometry of CR-submanifolds of a Kähler manifold was initiated by Bejancu and Chen and it has been developed by other authors [1–6, 12, 13].

The purpose of the present paper is to study CR-lightlike submanifolds of an indefinite Kähler product manifold which were defined in [7]. We have proved some properties of such submanifolds. We have also obtained necessary and sufficient conditions for CR-lightlike submanifolds in an indefinite Kähler product manifold to be either $D$-geodesic, $D'$-geodesic, CR-lightlike product, mixed-geodesic, totally-umbilical CR-lightlike, or curvature-invariant CR-lightlike submanifold.

Let $\bar{M}$ be a real $(m + n)$-dimensional semi-Riemannian manifold, $m, n > 1$ and $\bar{g}$ be a semi-Riemannian metric on $\bar{M}$. We denote by $q$ the constant index of $\bar{g}$ and suppose that $q > 0$. Now, let $M$ be a submanifold of codimension $n$ in $\bar{M}$. If the restriction $g = \bar{g}|_M$ of $\bar{g}$ to $M$ is still non-degenerate, then $(M, g)$ becomes a semi-Riemannian manifold and it can be studied as a submanifold of a semi-Riemannian manifold $\bar{M}$ [4, 9, 10]. A different situation appears when $g$ is degenerate, then $(M, g)$ is said to be a lightlike submanifold of $\bar{M}$. The geometry of lightlike submanifolds semi-Riemannian manifolds has been considered by many authors [7, 8].

2. CR-LIGHTLIKE SUBMANIFOLDS

Let $\bar{M}$ be an $m$-dimensional lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. In this case, there exists a smooth distribution on $\bar{M}$, called radical distribution such that

$$RadTM : M \rightarrow T_xM$$

$$x \rightarrow RadT_xM$$

$$RadT_xM = T_xM \cap T_xM^\perp, \text{ for each } x \in M$$

where

$$TM^\perp = \{V_x \in T_x\bar{M} : \bar{g}(V_x, W_x) = 0, \forall W_x \in T_xM\}$$

If rank of $RadTM$ is $r$ ($r > 0$), $M$ is called $r$-lightlike submanifold of $\bar{M}$. Moreover, there are four cases, as follows:

- Case 1. $0 < r < \min\{m, n\}$
- Case 2. $1 < r = n < m$
- Case 3. $1 < r = m < n$
- Case 4. $1 < r = m = n$. 

According to these cases, the submanifold is called $r$-lightlike, coisotropic, isotropic, and totally lightlike submanifold, respectively [7].

Now, let $M$ be an $r$-lightlike submanifold of $\bar{M}$. We consider the complementary distribution $S(TM)$ of $\text{Rad}TM$ on $TM$. Then we have the direct orthogonal sum

$$ TM = \text{Rad}TM \perp S(TM) $$

For the lightlike submanifold $M$, $TM^\perp$ is not complementary to $TM$ in $TM|_M$ since $\text{Rad}TM = TM \cap TM^\perp$ is a distribution on $M$ of rank $r > 0$. Now, we consider a complementary vector bundle $S(TM^\perp)$ of $\text{Rad}TM$ in $TM^\perp$. It follows that $S(TM^\perp)$ is also non-degenerate with respect to $\bar{g}$ and that $TM^\perp$ has the following orthogonal direct decomposition

$$ TM^\perp = \text{Rad}TM \perp S(TM^\perp) $$

We call $S(TM)$ and $S(TM^\perp)$ a screen distribution and a screen transversal vector bundle of $M$, respectively. As $S(TM)$ is a non-degenerate vector bundle of $T\bar{M}|_M$, we put

$$ T\bar{M}|_M = S(TM) \perp S(TM^\perp) $$

where $S(TM)^\perp$ is the complementary orthogonal vector bundle of $S(TM)$ in $T\bar{M}|_M$. Note that $S(TM^\perp)$ is a vector subbundle of $S(TM)^\perp$ and both are non-degenerate, thus we have the following orthogonal direct decomposition

$$ S(TM)^\perp = S(TM^\perp) \perp S(TM^\perp)^\perp $$

Now, we recall the following Theorem from [7] for later use.

**Theorem 2.1.** Let $(M, g, S(TM), S(TM^\perp))$ be an $r$-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $\ell tr(TM)$ called a lightlike transversal bundle of $\text{Rad}TM$ in $S(TM^\perp)^\perp$ and a basis of $\ell tr(TM)$ consists of smooth sections $\{N_1, N_2, ..., N_r\}$ of $S(TM^\perp)^\perp$ such that

$$ \bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad 1 \leq i, j \leq r $$

where $\{\xi_1, \xi_2, ..., \xi_r\}$ is a basis of $\text{Rad}TM$.

Now, we consider the vector bundle

$$ \ell tr(TM) = \ell tr(TM) \perp S(TM^\perp) $$

Thus we have

$$ T\bar{M} = TM \oplus \ell tr(TM) $$

$$ = S(TM) \perp S(TM^\perp) \perp \{\text{Rad}TM \oplus \ell tr(TM)\} $$

Let $\nabla$ be the Levi–Civita connection on $\bar{M}$. Then we have

$$ \nabla_X Y = \nabla_X Y + h(X, Y), \quad \text{for any } X, Y \in \Gamma(TM) $$

and
\[ \nabla_X V = -A_Y X + \nabla_X V \quad \text{for} \quad X \in \Gamma(TM) \quad \text{and} \quad V \in \Gamma(tr(M)) \] (9)

where \( \nabla \) is the linear connection on \( M \) and \( \Gamma(TM) \) denotes the set of differentiable vector fields on \( M \). We denote the Riemannian curvature tensors of \( M \) and \( \Gamma(TM) \) by \( R \) and \( \bar{R} \), respectively, we have
\[ \bar{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \] (10)

\[ + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \]

for any \( X,Y,Z \in \Gamma(TM) \), where the covariant derivative of \( h \) is defined by
\[ (\nabla_X h)(Y,Z) = \nabla_{X}h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \] (11)

for any \( X,Y,Z \in \Gamma(TM) \). Using the projectors \( L : tr(TM) \rightarrow \ell tr(TM) \) and \( S : tr(TM) \rightarrow S(TM^\perp) \) we have
\[ \nabla_X Y = \nabla_X Y + h^s(X,Y) \] (12)
\[ \nabla_X N = -A_N X + \nabla_X^s N + D^s(X,N) \] (13)

and
\[ \nabla_X W = -A_W X + \nabla_X^s W + D^s(X,W) \] (14)

for any \( X,Y \in \Gamma(TM) \), \( N \in \Gamma(\ell tr(TM)) \), and \( W \in \Gamma(S(TM^\perp)) \), where \( Nh(X,Y) = h^s(X,Y) \), \( Sh(X,Y) = h^s(X,Y) \), \( \nabla^s_X N \), \( D^s(X,W) \in \Gamma(\ell tr(TM)) \), \( \nabla^s_X W \), \( D^s(X,W) \in \Gamma(S(TM^\perp)) \), and \( \nabla_X Y, A_W X \in \Gamma(TM) \). By using (12), (13), and (14) we obtain
\[ g(h^s(X,Y), W) + g(Y, D^s(X,W)) = g(A_W X, Y) \] (15)

Let \( T \) be the projection morphism of \( TM \) on \( S(TM) \) with respect to the orthogonal decomposition of \( TM \). Then we have
\[ \nabla_X TY = \nabla^s_X TY + h^s(X,TY) \quad \text{for any} \quad X,Y \in \Gamma(TM) \] (16)

and
\[ \nabla_X \xi = -A^s_\xi X + \nabla^s_X \xi \quad \text{for any} \quad \xi \in \Gamma(RadTM) \] (17)

where \( h^s(X,TY) \), \( \nabla^s_X \xi \in \Gamma(RadTM) \) and \( A^s_\xi X, \nabla^s_X TY \in \Gamma(S(TM)) \). Thus we have the following equations
\[ g(h^s(X,TY), \xi) = g(A^s_\xi X, TY), \quad g(h^s(X,TY), N) = g(A_N X, TY) \] (18)
\[ g(A^s_\xi TX, TY) = g(TX, A^s_\xi TY), \quad A^s_\xi \xi = 0 \] (19)

and
\[ \bar{g}(A_N X, TY) = \bar{g}(N, \nabla_X TY) \] (20)
In general, the induced connection on $M$ is not a metric connection. Since $\nabla$ is a metric connection, $\nabla g$ is obtained from (12) and (13) as

$$(\nabla_X g)(Y, Z) = \ddot{g}(h^\ell(X, Y), Z) + \ddot{g}(h^\ell(X, Z), Y)$$  \hspace{1cm} (21)$$

for any $X, Y, Z \in \Gamma(TM)$ [7].

Let $(\bar{M}, \bar{I}, \ddot{g})$ be a real $2m$-dimensional indefinite Kähler manifold and $M$ be a real $n$-dimensional lightlike submanifold of $\bar{M}$. We say that $M$ is a CR-lightlike submanifold if the following two conditions are satisfied:

A. $\bar{J}(\text{Rad}TM)$ is a distribution on $M$ such that

$$\text{Rad}TM \cap \bar{J}(\text{Rad}TM) = \{0\}$$

B. There exist vector bundles $S(TM)$, $S(TM^\perp)$, $\elltr(TM)$, $D_o$ and $D^\prime$ over $M$ such that

$$S(TM) = \{\bar{J}(\text{Rad}TM) \oplus D^\prime\} \perp D_o, \quad \bar{J}(D_o) = D_o, \quad \bar{J}(D^\prime) = L_1 \perp L_2$$

where $D_o$ is a non-degenerate distribution on $M$, and $L_1$ and $L_2$ are vector subbundles of $\elltr(TM)$ and $S(TM^\perp)$, respectively.

From the above definition and by using (2) we obtain that the tangent bundle of a CR-lightlike submanifold is decomposed as follows

$$TM = D \oplus D^\prime$$

where

$$D = \text{Rad}TM \perp \bar{J}(\text{Rad}TM) \perp D_o$$  \hspace{1cm} (22)$$

(For details, we refer to [7].)

3. INDEFINITE KÄHLER PRODUCT MANIFOLDS

Let $(\bar{M}_1, \bar{J}_1, \ddot{g}_1)$ and $(\bar{M}_2, \bar{J}_2, \ddot{g}_2)$ be real $2m_1$ and $2m_2$-dimensional indefinite Kähler manifolds with constant indexes $q_1 > 0$ and $q_2 > 0$, respectively. Let $\bar{M}_1 \times \bar{M}_2$ be a semi-Riemannian product of the semi-Riemannian manifolds $\bar{M}_1$ and $\bar{M}_2$. We denote the projection mappings of $\Gamma(T(\bar{M}_1 \times \bar{M}_2))$ to $\Gamma(T\bar{M}_1)$ and $\Gamma(T\bar{M}_2)$ by $P$ and $Q$, respectively. Then we have

$$P^2 = P, \quad Q^2 = Q, \quad P + Q = I, \quad PQ = QP = 0$$

If we put $F = P - Q$, then we can easily see that $F^2 = I$. We define a semi-Riemannian metric of $\bar{M}_1 \times \bar{M}_2$ by

$$\ddot{g}(X, Y) = \ddot{g}_1(PX, PY) + \ddot{g}_2(QX, QY), \quad \text{for any} \quad X, Y \in \Gamma(T(\bar{M}_1 \times \bar{M}_2))$$

Orthonormal bases of $T_x\bar{M}_1$ and $T_y\bar{M}_2$ combine to give orthonormal bases of $T_{(x, y)}(\bar{M}_1 \times \bar{M}_2)$ for each $(x, y) \in \bar{M}_1 \times \bar{M}_2$. Thus the index of $\ddot{g}$ has constant value $q_1 + q_2$. Furthermore, $(\bar{M}_1 \times \bar{M}_2, \ddot{g})$ becomes a semi-Riemannian manifold with constant index $(q_1 + q_2)$.

If we define a mapping by $\bar{J} = \bar{J}_1 P + \bar{J}_2 Q$ of $T(\bar{M}_1 \times \bar{M}_2)$ to $T(\bar{M}_1 \times \bar{M}_2)$, then we can easily see that

$$\bar{J}^2 = -I, \quad \bar{J}_1 P = PJ, \quad \bar{J}_2 Q = QJ, \quad \text{and} \quad F\bar{J} = JF$$
Thus \( \bar{J} \) is an almost complex structure on \( \bar{M}_1 \times \bar{M}_2 \). Moreover, if \((\bar{M}_1, \bar{J}_1, \bar{g}_1)\) and \((\bar{M}_2, \bar{J}_2, \bar{g}_2)\) are both indefinite almost Hermitian manifolds, then we have

\[
\bar{g}(\bar{J}X, \bar{J}Y) = \bar{g}_1(\bar{J}_1PX, P\bar{J}Y) + \bar{g}_2(Q\bar{J}X, Q\bar{J}Y)
+ \bar{g}_2(\bar{J}_2QX, \bar{J}_2QY)
= \bar{g}_1(\bar{J}_1PX, \bar{J}_1PY) + \bar{g}(\bar{J}_2QX, \bar{J}_2QY)
= \bar{g}_1(\bar{P}X, \bar{P}Y) + \bar{g}_2(QX, QY) = \bar{g}(X, Y)
\]

for any \( X, Y \in \Gamma(T(\bar{M}_1 \times \bar{M}_2)) \). Thus \((\bar{M}_1 \times \bar{M}_2, \bar{J}, \bar{g})\) is an indefinite almost Hermitian manifold. If we denote the Levi–Civita connection on \( \bar{M}_1 \times \bar{M}_2 \) by \( \bar{\nabla} \), then by direct calculations, we obtain

\[
(\bar{\nabla}_X \bar{J})Y = (\bar{\nabla}_X \bar{J}_1)PY + (\bar{\nabla}_Q \bar{J}_2)QY + (\bar{\nabla}_Q \bar{J})PY + (\bar{\nabla}_P \bar{J})QY
\]

for any \( X, Y \in \Gamma(T(\bar{M}_1 \times \bar{M}_2)) \). If \((\bar{M}_1 \times \bar{M}_2, \bar{J}, \bar{g})\) is an indefinite Kähler manifold, then we have

\[
(\bar{\nabla}_P \bar{J}_1)PY + (\bar{\nabla}_Q \bar{J}_2)QY + (\bar{\nabla}_Q \bar{J})PY + (\bar{\nabla}_P \bar{J})QY = 0 \tag{23}
\]

Here taking \( FX \) instead of \( X \), we obtain

\[
(\bar{\nabla}_P \bar{J}_1)PY + (\bar{\nabla}_Q \bar{J}_2)QY - (\bar{\nabla}_Q \bar{J})PY - (\bar{\nabla}_P \bar{J})QY = 0 \tag{24}
\]

From Equations (23) and (24), we get

\[
(\bar{\nabla}_P \bar{J}_1)PY + (\bar{\nabla}_Q \bar{J}_2)QY = 0
\]

which implies that \((\bar{M}_1, \bar{J}_1, \bar{g}_1)\) and \((\bar{M}_2, \bar{J}_2, \bar{g}_2)\) are both indefinite Kähler manifolds. In the rest of this paper, we denote an indefinite Kähler product manifold by \((\bar{M}, \bar{J}, \bar{g})\).

If \( \bar{M}_1 \) and \( \bar{M}_2 \) are indefinite complex space forms with constant holomorphic sectional curvatures \( c_1, c_2 \) and we denote them by \( \bar{M}_1(c_1) \) and \( \bar{M}_2(c_2) \), respectively, then the Riemannian curvature tensor \( \bar{R} \) of Kähler product manifold \( \bar{M}_1(c_1) \times \bar{M}_2(c_2) \) is given by the formula

\[
\bar{R}(X, Y)Z = \frac{1}{16}(c_1 + c_2)[\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(\bar{J}Y, Z)\bar{J}X - \bar{g}(\bar{J}X, Z)\bar{J}Y
+ 2\bar{g}(X, \bar{J}Y)\bar{J}Z + 2\bar{g}(F\bar{J}X, Z)F\bar{J}Y - \bar{g}(F\bar{J}X, Z)F\bar{J}X
- \bar{g}(F\bar{J}X, Z)F\bar{J}Y + 2\bar{g}(F\bar{J}X, Y)F\bar{J}Z]
+ \frac{1}{16}(c_1 - c_2)[\bar{g}(F\bar{J}Y, Z)X - \bar{g}(F\bar{J}X, Z)Y + \bar{g}(\bar{J}Y, Z)\bar{J}X - \bar{g}(\bar{J}X, Z)\bar{J}Y
+ \bar{g}(\bar{J}Y, Z)\bar{J}X - \bar{g}(\bar{J}X, Z)\bar{J}Y + 2\bar{g}(F\bar{J}Y, Z)F\bar{J}Z]
+ 2\bar{g}(F\bar{J}Y, \bar{J}Z) + 2\bar{g}(X, \bar{J}Y)\bar{J}FZ \tag{25}
\]

for all \( X, Y, Z \in \Gamma(T(\bar{M}_1 \times \bar{M}_2)) \) [9].
4. CR-LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KÄHLER PRODUCT MANIFOLD

The following corollary is needed [7].

**Corollary 4.1.** Let \((M_1, g_1)\) and \((M_2, g_2)\) be a lightlike real hypersurface and a non-degenerate real hypersurface of the indefinite almost manifolds \((\tilde{M}_1, \tilde{J}_1, \tilde{g}_1)\) and \((\tilde{M}_2, \tilde{J}_2, \tilde{g}_2)\), respectively. Then \((M_1 \times M_2, g_1 \times g_2)\) is a CR-lightlike submanifold of codimension 2 of \((\tilde{M}_1 \times \tilde{M}_2, \tilde{J}_1 \times \tilde{J}_2, \tilde{g}_1 \times \tilde{g}_2)\).

**Example 4.2.** We identify \(C_q^m\) with \((\mathbb{R}^{2m}_{2q}, \tilde{J}, \tilde{g})\), where \(\tilde{J}\) and \(\tilde{g}\) are, respectively, defined by

\[
\tilde{J}(x^1, y^1, ..., x^m, y^m) = (-y^1, x^1, ..., -y^m, x^m)
\]

and

\[
\tilde{g}((x^1, y^1, ..., x^m, y^m), (u^1, v^1, ..., u^m, v^m)) = -\sum_{i=1}^{q}\{x^i u^i + y^i v^i\} + \sum_{j=q+1}^{m}\{x^j u^j + y^j v^j\}
\]

Now, Let \(\mathbb{R}^{2m}_{2q}\) and \(\mathbb{R}^{2n}_{2s}\) be two indefinite Kähler flat spaces. The lightlike cone \(\bigwedge^{2m-1}_{2q-1}\) of \(\mathbb{R}^{2m}_{2q}\) given by the equation

\[
\sum_{i=1}^{q}\{(x^i)^2 + (y^i)^2\} - \sum_{j=q+1}^{m}\{(x^j)^2 + (y^j)^2\} = 0, \quad X \neq 0
\]

is a lightlike hypersurface of \(\mathbb{R}^{2m}_{2q}\). Moreover, consider the pseudosphere \(S^{2n-1}_{2s}(r)\) of \(\mathbb{R}^{2n}_{2s}\) given by

\[
S^{2n-1}_{2s}(r) = \{X \in \mathbb{R}^{2n}_{2s}; -\sum_{i=1}^{s}\{(x^i)^2 + (y^i)^2\} + \sum_{j=s+1}^{n}\{(x^j)^2 + (y^j)^2\} = r^2\}
\]

where \(X = (x^1, y^1, ..., x^n, y^n) \in \mathbb{R}^{2n}\). Then from Corollary 4.1, \(\bigwedge^{2m-1}_{2q-1} \times S^{2n-1}_{2s}(r)\) is a CR-lightlike submanifold of codimension 2 of \(\mathbb{R}^{2(n+m)}_{2(s+q)}\).

Let \(M\) be a CR-lightlike submanifold of an indefinite Kähler product manifold \((\tilde{M}, \tilde{J}, \tilde{g})\). If we denote by \(R\) and \(S\) the projections on \(D\) and \(D^\perp\), respectively, then we have

\[
\tilde{J}X = fX + \omega X, \quad \text{for any } X \in \Gamma(TM)
\]  

(26)

where \(fX = \tilde{J}RX\) and \(\omega X = \tilde{J}SX\). Clearly, \(f\) is a tensor field of type \((1,1)\) and \(\omega\) is a \(\Gamma(L_1 \perp L_2)\)-valued 1-form on \(M\). On the other hand, we set

\[
\tilde{J}V = BV + CV, \quad \text{for any } V \in \Gamma(tr(TM))
\]  

(27)

where \(BV\) and \(CV\) are sections of \(\Gamma(TM)\) and \(\Gamma(tr(TM))\), respectively.

**Theorem 4.3.** Let \(M\) be a coisotropic submanifold of an indefinite Kähler product manifold \(\tilde{M}_1(c_1) \times \tilde{M}_2(c_2)\) with \(c_1, c_2 \neq 0\). Then \(M\) is a CR-lightlike coisotropic submanifold with \(D_0 \neq \{0\}\) and \(D' = \tilde{J}(tr(TM))\) if and only if the following conditions are satisfied:
(i) The maximal complex subspaces of $T_xM$, $x \in M$, define a distribution

$$ D = TM^+ \perp \bar{J}(TM^+) \perp D_o $$

where $D_o$ is a non-degenerate non-zero almost complex distribution on $M$.

(ii) There exists a lightlike transversal vector bundle $\ell tr(TM)$ such that the Riemannian curvature tensor of $M_1(c_1) \times M_2(c_2)$ satisfies

$$ \bar{g}(\bar{R}(X,Y)N, N') = 0 $$

for any $X,Y \in \Gamma(D_o)$ and $N,N' \in \Gamma(\ell tr(TM))$.

**Proof.** Let us assume that $M$ is a CR-lightlike coisotropic submanifold of an indefinite $M_1(c_1) \times M_2(c_2)$ with $D_o \neq 0$, $c_1, c_2 \neq 0$, and $D' = \bar{J}(\ell tr(TM))$. Since $\text{Rad}TM = TM^+$, from (22) we get

$$ TM \cap \bar{J}(TM) = D = TM^+ \perp \bar{J}(TM^+) \perp D_o $$

and $D_o$ satisfies the condition (i). From Equation (25) we have

$$ \bar{R}(X,Y)N = \frac{1}{8}(c_1 + c_2)\{\bar{g}(X,JY)\bar{J}N + \bar{g}(FX,JY)F\bar{J}N\} $$

$$ + \frac{1}{8}(c_1 - c_2)\{\bar{g}(FX,JY)\bar{J}N + \bar{g}(X,JY)F\bar{J}N\} $$

for all $X,Y \in \Gamma(D_o)$ and $N \in \Gamma(\ell tr(TM))$. Take an $\ell tr(TM)$ corresponding to the screen distribution $S(TM)$ from condition (B), by using (25) and taking into account that $D_o$ is an almost complex distribution and $\ell tr(TM)$ is orthogonal to both $D'$ and $D_o$, we obtain (28).

Conversely, let us assume that conditions (i) and (ii) are satisfied. Then from (i), we conclude that $\bar{J}(TM^+)$ is a distribution on $M$ such that $TM^+ \cap \bar{J}(TM^+) = \{0\}$, that is, the condition (A) is satisfied. Now, we choose $S(TM)$ such that it contains $\bar{J}(TM^+) \perp D_o$ and consider the corresponding lightlike transversal vector bundle $\ell tr(TM)$. Taking into account that $\ell tr(TM)$ is orthogonal to $\bar{J}(TM^+)$, we conclude

$$ \bar{g}(\bar{J}N, \xi) = -\bar{g}(N, \xi) = 0, \forall N \in \Gamma(\ell tr(TM)) \text{ and } \xi \in \Gamma(TM^+) $$

Thus $\bar{J}(\ell tr(TM))$ is a distribution on $M$. Moreover, by using (25) and (28) we obtain

$$ \bar{g}(X,JY)\{(c_1 + c_2)\bar{g}(\bar{J}N, N') + (c_1 - c_2)\bar{g}(F\bar{J}N, N')\} $$

$$ \bar{g}(FX,JY)\{(c_1 + c_2)\bar{g}(F\bar{J}N, N') + (c_1 - c_2)\bar{g}(\bar{J}N, N')\} = 0 $$

for any $X,Y \in \Gamma(D_o)$ and $N, N' \in \Gamma(\ell tr(TM))$. As vector fields $X$ and $FX$ are independent, $c_1, c_2 \neq 0$ and $D_o \neq 0$, it follows that

$$ 4c_1c_2\bar{g}(\bar{J}N, N') = 0 $$

Thus $\bar{J}(\ell tr(TM)) \cap TM^+ = \{0\}$. Since $\bar{J}(\ell tr(TM))$ is orthogonal to $TM^+ \oplus \ell tr(TM)$, from Equation (7) we obtain $D' = \bar{J}(\ell tr(TM))$ is a vector subbundle of $S(TM)$. On the other hand, $D' \cap \bar{J}(TM^+) = \{0\}$ and $D'$ is orthogonal to $D_o$, we obtain

$$ S(TM) = \{\bar{J}(TM^+) \oplus D'\} \perp D_o \perp D_1 $$
where we have to show that \( D_1 = \{0\} \). First, by direct calculations, it follows that \( D_1 \) is an almost complex distribution. Since \( D_1 \cap D = \{0\} \) and \( D \) is the maximal almost complex distribution on \( M \), we conclude \( D_1 = \{0\} \). As \( J(D^\perp) = \ell tr(TM) \), the condition (B) is also satisfied. Thus the proof is complete. \( \square \)

From the general theory of semi-invariant submanifolds of Kähler manifolds, we have the following theorem [7].

**Theorem 4.4.** Let \( M \) be a CR-lightlike submanifold of an indefinite Kähler product manifold \( \bar{M}_1 \times \bar{M}_2 \). Then we have the following assertions

(i) The almost complex distribution \( D \) is integrable if and only if the second fundamental form of \( M \) satisfies

\[
h(X, JY) = h(JX, Y)
\]

for any \( X, Y \in \Gamma(D) \).

(ii) The totally real distribution \( D' \) is integrable if and only if the shape operator of \( M \) satisfies

\[
A_{jZ}U = A_{jU}Z
\]

for any \( Z, U \in \Gamma(D') \).

Let \( M \) be a CR-lightlike submanifold of an indefinite Kähler product manifold \( \bar{M}_1 \times \bar{M}_2 \). \( M \) is said to be a mixed-geodesic submanifold if the second fundamental form of \( M \) satisfies \( h(X, Z) = 0 \) for any \( X \in \Gamma(D) \) and \( Z \in \Gamma(D') \).

**Theorem 4.5.** Let \( M \) be a proper CR-lightlike submanifold of an indefinite Kähler product manifold \( \bar{M}_1 \times \bar{M}_2 \). \( M \) is a mixed-geodesic CR-lightlike submanifold if and only if the following conditions are satisfied:

(1) \( A^*_\xi X \) has no component in \( \Gamma(\bar{J}L_2 \perp \bar{J}RadTM) \)

(2) \( A_W X \) has no component in \( \Gamma(\bar{J}L_2 \perp \bar{J}RadTM) \)

for any \( X \in \Gamma(D_\alpha), \xi \in \Gamma(RadTM) \) and \( W \in \Gamma(S(TM^\perp)) \).

**Proof.** We have to show that \( \bar{g}(h(X, Z), \xi) = \bar{g}(h(X, Z), W) = 0 \), for any \( X \in \Gamma(D), Z \in \Gamma(D'), \xi \in \Gamma(RadTM) \) and \( W \in \Gamma(S(TM^\perp)) \). Thus by using (12), (13), and (16) we have

\[
\bar{g}(h(X, Z), \xi) = \bar{g}(\bar{\nabla}_X Z, \xi) = -\bar{g}(\bar{\nabla}_X \xi, Z) = -\bar{g}(\nabla_X \xi, Z)
\]

and

\[
\bar{g}(h(X, Z), W) = \bar{g}(\bar{\nabla}_X Z, W) = -\bar{g}(\bar{\nabla}_X W, Z) = \bar{g}(A_W X, Z)
\]

for any \( X \in \Gamma(D_\alpha), Z \in \Gamma(D'), \xi \in \Gamma(RadTM), \) and \( W \in \Gamma(S(TM^\perp)) \). Thus \( M \) is a mixed-geodesic submanifold if and only if the conditions (1) and (2) are satisfied. \( \square \)

Let \( M \) be a proper CR-lightlike submanifold of an indefinite Kähler product manifold \( \bar{M}_1 \times \bar{M}_2 \). \( M \) is said to be a \( D \)-geodesic (resp. \( D' \)-geodesic) submanifold if the second fundamental form of \( M \) satisfies \( h(X, Y) = 0 \) (resp. \( h(Z, W) = 0 \)) for any \( X, Y \in \Gamma(D) \) (resp. \( Z, W \in \Gamma(D') \)).
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Theorem 4.6. Let $M$ be a proper CR-lightlike submanifold of an indefinite Kähler product manifold $\bar{M}_1 \times \bar{M}_2$. $M$ is a $D$-geodesic CR-lightlike submanifold if and only if the following conditions are satisfied:

1. $A^*_\xi X$ has only component in $\Gamma(\bar{J}RadTM \oplus \bar{J}L_1)$

2. $\bar{g}(A_W X, Y) = \bar{g}(D^f(X, W), Y)$
   for any $X, Y \in \Gamma(D)$, $W \in \Gamma(S(TM)^{\perp})$ and $\xi \in \Gamma(RadTM)$.

Proof. From (12), (17), and taking into account that $\bar{\nabla}$ is a Levi–Civita connection, we have

$$\bar{g}(h(X, Y), \xi) = \bar{g}(\nabla_X Y, \xi) = X \bar{g}(Y, \xi) - \bar{g}(\nabla_X \xi, Y)$$

$$= -\bar{g}(\nabla_X Y, \xi) - \bar{g}(h(X, \xi), Y)$$

$$= \bar{g}(A^*_\xi X, Y) - \bar{g}(\nabla_X \xi, \xi)$$

$$= \bar{g}(A^*_\xi X, Y)$$

and by using (14) we get

$$\bar{g}(h(X, Y), W) = \bar{g}(A_W X, Y) - \bar{g}(h(X, Y), W)$$

which proves our assertion, for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(RadTM)$, and $W \in \Gamma(S(TM)^{\perp})$, where $Y = \xi + J\xi + Y_o \in \Gamma(RadTM \perp JRadTM \perp D_o)$. □

Theorem 4.7. Let $M$ be a proper CR-lightlike submanifold of an indefinite Kähler product manifold $\bar{M}_1 \times \bar{M}_2$. $M$ is a $D^f$-geodesic CR-lightlike submanifold if and only if the following conditions are satisfied:

1. $A^*_\xi Z$ has no component in $\Gamma(\bar{J}L_2 \perp JRadTM)$

2. $A_W Z$ has only component in $\Gamma(\{RadTM \perp D_o\} \oplus J\bar{L}_1)$,
   for any $Z \in \Gamma(D^f)$, $\xi \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM)^{\perp})$.

Proof. By using (12), (14), (16), and taking into account that $\bar{\nabla}$ is a Levi–Civita connection, by direct calculations, we obtain

$$\bar{g}(h(Y, Z), \xi) = \bar{g}(A^*_\xi Z, Y)$$

and

$$\bar{g}(h(Y, Z), W) = \bar{g}(\nabla_Z Y, W) = Z \bar{g}(Y, W) - \bar{g}(\nabla_Z W, Y)$$

$$= -\bar{g}(\nabla_Z W, Y) - \bar{g}(A_W Z, Y) - \bar{g}(\nabla^*_Z W, Y) - \bar{g}(D^f(Z, W), Y)$$

$$= \bar{g}(A_W Z, Y)$$

for any $Z, Y \in \Gamma(D^f)$, $\xi \in \Gamma(RadTM)$, and $W \in \Gamma(S(TM)^{\perp})$. Thus the proof is complete. □

Next, we recall the following general theory from [7] for later use.

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Let $M$ be a proper CR-lightlike submanifold of a Kähler manifold $\tilde{M}$ such that $D$ and $D'$ define totally geodesic foliations on $M$. In this case, $M$ is locally represented as a product manifold $M_1 \times M_2$, where $M_1$ and $M_2$ are integral manifolds leaves of $D$ and $D'$, respectively, and they both are totally geodesic immersed in $M$. Then $M$ is said to be a CR-lightlike product.

**Theorem 4.8.** Let $M$ be a proper CR-lightlike submanifold of an indefinite Kähler product manifold $\tilde{M}_1 \times \tilde{M}_2$. $M$ is a CR-lightlike product if and only if $(\nabla_X f)Y = 0$ for any $X, Y \in \Gamma(TM)$.

**Proof.** Let us assume that $M$ is a CR-lightlike product. Then the distributions $D$ and $D'$ are integrable and their leaves are totally geodesic in $M$. By using (8), (26), and (27) we get

$$\bar{\nabla}_Z fX + h(Z, fX) = \bar{J}\nabla_Z X + \bar{J}h(Z, X)$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(TM)$. Thus we have

$$\nabla_Z fX = f\nabla_Z X + Bh(Z, X),$$

that is, $(\nabla_Z f)X = Bh(Z, X)$ (30)

where $(\nabla_Z f)X \in \Gamma(D)$ and $Bh(Z, X) \in \Gamma(D')$. Since $D$ is an invariant distribution and $M$ is a CR-lightlike product, we have

$$\bar{g}(Bh(Z, X), Y) = \bar{g}(\bar{J}h(Z, X), Y) - \bar{g}(Ch(Z, X), Y)$$

$$= -\bar{g}(h(Z, X), \bar{J}Y) - \bar{g}(Ch(Z, X), Y)$$

$$= -\bar{g}(h(Z, X), \bar{J}\xi - \xi + \bar{J}Y_0) - \bar{g}(Ch(Z, X), \xi + \bar{J}\xi + Y_0)$$

$$= \bar{g}(h(Z, X), \xi) - \bar{g}(Ch(Z, X), \xi)$$

$$= \bar{g}(h(Z, X), \xi) - \bar{g}(\bar{J}Ch(Z, X), \bar{J}\xi)$$

$$= \bar{g}(\nabla_Z X, \xi) = \bar{g}(\nabla_Z JX, \bar{J}\xi)$$

$$= \bar{g}(\nabla_Z fX, \bar{J}\xi) = 0$$

for any $Y \in \Gamma(D)$. Thus we conclude that $Bh(Z, X)$ has no component in $\Gamma(D)$, which implies that $(\nabla_Z f)X = 0$, where $Y = \xi + \bar{J}\xi + Y_0 \in \Gamma(RadTM \perp \bar{J}RadTM \perp D_0)$.

Since $M$ is a CR-lightlike product, we have $\nabla_Z W \in \Gamma(D')$, for any $W \in \Gamma(D')$ and $Z \in \Gamma(TM)$. Thus we get

$$(\nabla_Z f)W = \nabla_Z fW - f(\nabla_Z W) = 0$$

Conversely, we suppose that $\nabla f = 0$. Then we have

$$f\nabla_Y X = \nabla_Y fX \in \Gamma(D), \ \text{for any } X, Y \in \Gamma(D)$$
and
\[ f\nabla_Z W = \nabla_Z fW = 0, \quad \text{for any } Z, W \in \Gamma(D') \]
which implies that \( \nabla_Z W \in \Gamma(D') \). Thus we have the leaves of distributions \( D \) and \( D' \) are totally geodesic submanifolds in \( M \). This completes the proof. \( \square \)

**Theorem 4.9.** Let \( M \) be a proper CR-lightlike submanifold of an indefinite Kähler product manifold \( \hat{M}_1 \times \hat{M}_2 \). \( M \) is a CR-lightlike product if and only if \( Bh(Z, X) = 0 \), for any \( Z \in \Gamma(TM) \) and \( X \in \Gamma(D) \).

**Proof.** We suppose that \( M \) is a CR-lightlike product. Then we conclude that both distributions \( D \) and \( D' \) are integrable and their leaves are totally geodesic submanifolds in \( M \). From (30) we have
\[ Bh(Z, X) = 0, \quad \text{for any } Z \in \Gamma(TM), \quad X \in \Gamma(D) \]
Conversely, if \( Bh(Z, X) = 0 \), then from (30) we get
\[ (\nabla_Z f)X = 0, \quad \text{for any } Z \in \Gamma(TM), \quad X \in \Gamma(D) \]
which implies that \( D \) is totally geodesic in \( M \). Moreover, we conclude
\[ f\nabla_Z W = \nabla_Z fW = 0, \quad \text{for any } Z \in \Gamma(TM), \quad W \in \Gamma(D') \]
which implies that \( \nabla_Z W \in \Gamma(D') \). Thus \( D' \) is also totally geodesic in \( M \), which proves our assertion. \( \square \)

As a consequence of Theorem 4.8 and Theorem 4.9 we have

**Theorem 4.10.** Let \( M \) be a proper CR-lightlike submanifold of an indefinite Kähler product manifold \( \hat{M}_1 \times \hat{M}_2 \). Then \( M \) is a CR-lightlike product if \( M \) is a totally geodesic lightlike submanifold of an indefinite Kähler product manifold \( \hat{M}_1 \times \hat{M}_2 \).

Taking into account that the curvature tensor field of \( \hat{M}_1(c_1) \times \hat{M}_2(c_2) \) is given by (25), we have special forms for the structure equations of Gauss and Codazzi for the immersion of a CR-lightlike submanifold \( M \) in \( \hat{M}_1(c_1) \times \hat{M}_2(c_2) \). Thus the equation of Gauss becomes
\[
R(X, Y)Z = \frac{1}{16}(c_1 + c_2)\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(\bar{J}Y, Z)fX \\
- \bar{g}(\bar{J}X, Z)fY + 2\bar{g}(X, \bar{J}Y)fZ + 2\bar{g}(FY, Z)FX - \bar{g}(FX, Z)FY \\
+ \bar{g}(F\bar{J}Y, Z)FfX - \bar{g}(F\bar{J}X, Z)FfY + 2\bar{g}(FX, \bar{J}Y)FfZ \} \\
+ \frac{1}{16}(c_1 - c_2)\{\bar{g}(FY, Z)X - \bar{g}(FX, Z)Y + \bar{g}(Y, Z)FX - \bar{g}(X, Z)FY \\
+ \bar{g}(F\bar{J}Y, Z)fX - \bar{g}(F\bar{J}X, Z)fY + \bar{g}(\bar{J}Y, Z)FFX \\
- \bar{g}(\bar{J}X, Z)FFY + 2\bar{g}(FX, \bar{J}Y)FFZ + 2\bar{g}(X, \bar{J}Y)FFZ \}
\]
\[ + A_{h(Y, Z)}X - A_{h(X, Z)}Y \tag{31} \]
for all $X,Y,Z$ tangent to $M$, where $R$ is the curvature tensor of $M$. The equation of Codazzi is given by

$$ (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) = \frac{1}{16}(c_1 + c_2)\{\bar{g}(JY,Z)\omega X - \bar{g}(JX,Z)\omega Y + 2\bar{g}(FX,JY)F\omega Z + \bar{g}(FJX,Z)F\omega Y \\
+ 2\bar{g}(FX,JY)F\omega Z\} + \frac{1}{16}(c_1 - c_2)\{\bar{g}(FJY,Z)\omega X - \bar{g}(FJX,Z)\omega Y + 2\bar{g}(FJX,JY)\omega Z - \bar{g}(FJX,Z)F\omega Y \\
+ 2\bar{g}(FX,JY)\omega Z\} $$

\[ (32) \]

for all $X,Y,Z$ tangent to $M$, where $h$ is the second fundamental of $M$.

**Theorem 4.11.** Let $M$ be a proper CR-lightlike submanifold of an indefinite Kähler product manifold $\tilde{M}_1 \times \tilde{M}_2$. There exist no proper curvature-invariant CR-lightlike submanifolds in $\tilde{M}_1(c_1) \times \tilde{M}_2(c_2)$ with $c_1,c_2 \neq 0$.

**Proof.** We suppose that $M$ is a proper curvature-invariant CR-lightlike submanifold of $\tilde{M}_1(c_1) \times \tilde{M}_2(c_2)$. By using (32) we have

$$ (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) = \frac{1}{8}(c_1 + c_2)\{\bar{g}(JY,Z)\omega X + \bar{g}(FX,JY)F\omega Z\} + \frac{1}{8}(c_1 - c_2)\{\bar{g}(FX,JY)\omega X + \bar{g}(FJX,Z)\omega Y + 2\bar{g}(FJX,JY)\omega Z + \bar{g}(FJX,Z)F\omega Y \\
+ 2\bar{g}(FX,JY)F\omega Z\} $$

\[ (33) \]

for any $X,Y \in \Gamma(D_o)$ and $Z \in \Gamma(D')$, which implies that

$$ \bar{g}(X,JY)\{(c_1 + c_2)\omega Z + (c_1 - c_2)F\omega Z\} \\
+ \bar{g}(FX,JY)\{(c_1 + c_2)F\omega Z + (c_1 - c_2)\omega Z\} = 0. $$

Since vector fields $X$ and $FX$ are independent, we conclude

$$ (c_1 + c_2)\omega Z + (c_1 - c_2)F\omega Z = 0 $$

and

$$ (c_1 + c_2)F\omega Z + (c_1 - c_2)\omega Z = 0. $$

This implies that

$$ 4c_1c_2\omega Z = 0 $$

As $D' \neq \{0\}$ and $c_1,c_2 \neq 0$, this is impossible. Thus the proof is complete. \qed

We have the following result for a totally geodesic submanifold to be a curvature-invariant submanifold.
Theorem 4.12. Let $M$ be a proper CR-lightlike submanifold of an indefinite Kähler product manifold $\bar{M}_1 \times \bar{M}_2$. Then there exist no proper totally geodesic CR-lightlike submanifolds in $\bar{M}_1(c_1) \times \bar{M}_2(c_2)$ with $c_1, c_2 \neq 0$.

Theorem 4.13. Let $M$ be a proper CR-lightlike submanifold of an indefinite Kähler product manifold $\bar{M}_1 \times \bar{M}_2$. Then there exist no proper totally umbilical CR-lightlike submanifolds in $\bar{M}_1(c_1) \times \bar{M}_2(c_2)$ with $c_1 + c_2 \neq 0$.

Proof. We suppose that $M$ is a proper totally umbilical CR-lightlike submanifold of an indefinite Kähler product manifold $\bar{M}_1(c_1) \times \bar{M}_2(c_2)$, then there is a smooth transversal vector field $H \in \Gamma(tr(TM))$, called transversal vector field of $M$ in $\bar{M}_1 \times \bar{M}_2$, such that

$$h(X, Y) = \bar{g}(X, Y)H, \quad \text{for any } X, Y \in \Gamma(TM) \quad (34)$$

By using the Equations (11) and (34), we obtain

$$(\nabla_X h)(Y, Z) = \bar{g}(Y, Z)\nabla_X^⊥ H \quad (35)$$

for any $X, Y \in \Gamma(TM)$. From the Gauss formulae, we have

$$\bar{g}(\bar{R}(Z, X)JX, JZ) = \bar{g}((\nabla_Z h)(X, JX), JZ) - \bar{g}((\nabla_X h)(Z, JX), JZ)$$

$$\quad = \bar{g}(X, JX)\bar{g}(\nabla_X^⊥ H, JZ) - \bar{g}(Z, JX)\bar{g}(\nabla_X^⊥ H, JZ)$$

$$\quad = 0 \quad (36)$$

for any $X \in \Gamma(D_o)$ and $Z \in \Gamma(JL_2)$. Moreover, by using (25) we obtain

$$\bar{g}(\bar{R}(Z, X)X, Z) = \frac{1}{16}(c_1 + c_2)(1 + 2\bar{g}(X, FX)\bar{g}(FZ, Z))$$

$$\quad + \frac{1}{16}(c_1 - c_2)(\bar{g}(FX, X) + \bar{g}(FZ, Z)) \quad (37)$$

for any orthonormal vector fields $X \in \Gamma(D_o)$ and $Z \in \Gamma(D)$. As $X, FX$ and $Z, FZ$ are independent, they can be chosen as orthogonal vector fields. In this case, we get

$$\bar{g}(\bar{R}(Z, X)X, Z) = \frac{1}{16}(c_1 + c_2)$$

This is a contradiction. This completes the proof. \hfill \Box

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