GLOBAL EXPONENTIAL STABILITY FOR REACTION–DIFFUSION RECURRENT NEURAL NETWORKS WITH MULTIPLE TIME-VARYING DELAYS

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الخلاصة

سوف نعرض في هذا البحث الثبات الأسّي للشبكات العصبية ذات التأخر الزمني المتغير والمتعدد وحدود التفاعلية الانتشارية. ويفترض أن اقترانات التنشيط تكون محدودة ومتمصلة كلياً وفقاً بـ (Lipschitz).

وقد حصلنا على الشروط الكافية باستخدام اقترانات (Lyapunov) التي تضمن الثبات الأسّي الكلي للشبكة العصبية المعاقّة. سوف نورد مثالاً حسابياً لإظهار مدى صحة طريقة التحليل التي اتبعناها.

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ABSTRACT

In this paper, we consider the problem of exponential stability for recurrent neural networks with multiple time-varying delays and reaction–diffusion terms. The activation functions are supposed to be bounded and globally Lipschitz continuous. By means of Lyapunov functionals, sufficient conditions are derived, which guarantee global exponential stability of the delayed neural network. Finally, a numerical example is given to show the correctness of our analysis.

Key words: Global exponential stability; reaction-diffusion terms; neural networks; multiple time-varying delays; Lyapunov functional
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1. INTRODUCTION

In applications of neural networks without or with delays to some practical problems, such as optimization solvers [1], pattern recognition, image compression [2], and quadratic programming problems [3,4], the stability properties of the system play an important role. Stability analysis of neural networks has received wide attention in recent years, and various types of stability conditions have been proposed in the literature (see, e.g., [5–12], and the references therein).

Diffusion effects can arise in neural networks models such as when electrons are moving in asymmetric electromagnetic fields; so we must consider both space and time variations. Recently, many efforts have been devoted to the study of neural networks with diffusion terms [13–15, 25–26], which are expressed by partial differential equations. Moreover, stability of neural networks with multiple time-varying delays have also been studied in [16–19, 27]. However, to the best of our knowledge, few authors consider global exponential stability for the recurrent neural networks with multiple time-varying delays and reaction–diffusion terms.

In this paper, we propose some criteria for the global exponential stability of recurrent neural networks with multiple time-varying delays and reaction–diffusion terms by constructing suitable Lyapunov functional and using some analytic techniques. The activation functions are supposed to be bounded and globally Lipschitz continuous, which are more general than the usual bounded monotonically increasing ones such as the activation functions of the sigmoidal type. The results may have significant impact on the design and applications of globally exponentially stable reaction–diffusion recurrent neural networks with multiple time-varying delays, and are of great interest in many applications.

2. MODEL DESCRIPTION AND PRELIMINARIES

Consider the following neural network with multiple time-varying delays and reaction–diffusion terms:

\[
\frac{\partial u_i(t, x)}{\partial t} = \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - a_i u_i(t, x) \\
+ \sum_{j=1}^{n} b_{ij} f_j(u_j(t, x)) + \sum_{m=1}^{r} \sum_{j=1}^{n} c_{ij}^{(m)} f_j(u_j(t - \tau_m(t), x)) + I_i
\]

for \( i \in \{1, 2, \cdots, n\}, t > 0, \) where \( x = (x_1, x_2, \cdots, x_l)^T \in \Omega \subset R^l, \) \( \Omega \) is a bounded compact set with smooth boundary \( \partial \Omega \) and \( \text{mes} \Omega > 0 \) in space \( R^l; \) \( u_i(t, x) \) is the state of the \( i \)th unit at time \( t; \) \( f_i(\cdot) \) denote the signal functions of the \( i \)th neurons at time \( t \) and in space \( x; \) \( I_i \) denote the external inputs on the \( i \)th neurons; \( a_i > 0 \) are constants; \( \tau_m(t), m = 1, \cdots, r, \) are time-varying delays of the neural network satisfying \( 0 \leq \tau_m(t) \leq \tau_m, \) \( 0 \leq \dot{\tau}_m(t) = d < 1 \) for \( m = 1, \cdots, r, \) and \( \sigma_m = \max_{1 \leq m \leq r} \{ \tau_m \}; \) \( b_{ij} \) and \( c_{ij}^{(m)} \) stand for the weights of neuron interconnections. Smooth functions \( D_{ik} = D_{ik}(t, x, u) \geq 0 \) correspond to the transmission diffusion operators along the \( i \)th neurons.
The boundary conditions and initial conditions are given by

$$\frac{\partial u_i}{\partial N} := \left( \frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, \ldots, \frac{\partial u_i}{\partial x_l} \right)^T = 0, \quad i = 1, 2, \ldots, n \quad (2)$$

and

$$u_i(s, x) = \phi_{ui}(s, x), \quad s \in [-\sigma_m, 0], \quad i = 1, 2, \ldots, n \quad (3)$$

where $N$ is the unit exterior normal vector to $\partial \Omega$, $\phi_{ui}(s, x)$ ($i = 1, 2, \ldots, n$) are bounded and continuous on $[-\sigma_m, 0] \times \Omega$.

We assume that the activation functions satisfy the following properties:

(H1) The neurons activation functions $f_i(\cdot)$ ($i = 1, 2, \ldots, n$) are bounded and Lipschitz-continuous, that is, there exist constants $F_i > 0$ such that

$$|f_i(\xi_1) - f_i(\xi_2)| \leq F_i |\xi_1 - \xi_2|$$

for all $\xi_1, \xi_2 \in R$.

(H2) $|f_i(\xi)| \leq L_i, \xi \in R^n$, and $L_i > 0, j = 1, 2, \ldots, n$.

For convenience, we introduce some notations. Let $u_i = u_i(t, x)$ and $u^* = (u^*_1, u^*_2, \ldots, u^*_n)^T$ be the equilibrium of system (1). For any $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, $||x||_1$ denotes 1-norm of $x$, i.e., $||x||_1 = \sum_{k=1}^{n} |x_k|$; for any $u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_m(t, x))^T \in R^m$, define

$$||u_i(t, x)||_2 = \left[ \int_{\Omega} |u_i(t, x)|^2 dx \right]^{1/2}, \quad i = 1, 2, \ldots, n$$

**Definition 1.** The equilibrium point $u^*$ of the system (1) is said to be globally exponentially stable, if there exists a constant $M \geq 0$ such that

$$\sum_{i=1}^{n} ||u_i - u^*_i||_2^2 \leq M e^{-2\varepsilon t} ||\phi_{u} - u^*||_2^2, \quad (4)$$

for all $t \geq 0$, where $u_i(t, x)$ ($i = 1, 2, \ldots, n$) is any solution of the system (1), and

$$||\phi_{u} - u^*||_2^2 = \sup_{s \in [-\sigma_m, 0]} \sum_{i=1}^{n} ||\phi_{ui}(s, x) - u^*_i||_2^2.$$  

(5)
3. MAIN RESULTS

In this section, we discuss the global exponential stability of the system (1) with the initial conditions (2–3) and give our main results.

Suppose \((u_1(t, x), u_2(t, x), \ldots, u_n(t, x))^T\) is any solution of the system (1), rewrite the system (1) as follows:

\[
\begin{align*}
\frac{\partial (u_i - u_i^*)}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial (u_i - u_i^*)}{\partial x_k} \right) - a_i(u_i - u_i^*) \\
&\quad + \sum_{j=1}^{n} b_{ij} \left[ f_j(u_j(t, x)) - f_j(u_j^*) \right] \\
&\quad + \sum_{m=1}^{r} \sum_{j=1}^{n} c_{ij}^{(m)} \left[ f_j(u_j(t - \tau_m(t), x)) - f_j(u_j^*) \right].
\end{align*}
\]

Multiply both sides of (6) by \(u_i - u_i^*\), and make an integral

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u_i - u_i^*)^2 \, dx = \sum_{k=1}^{l} \int_{\Omega} (u_i - u_i^*) \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial (u_i - u_i^*)}{\partial x_k} \right) \, dx \\
- \int_{\Omega} a_i(u_i - u_i^*)^2 \, dx + \sum_{j=1}^{n} \int_{\Omega} (u_i - u_i^*) b_{ij} (f_j(u_j) - f_j(u_j^*)) \, dx \\
+ \sum_{j=1}^{n} \int_{\Omega} (u_i - u_i^*) \sum_{m=1}^{r} c_{ij}^{(m)} \left[ f_j(u_j(t - \tau_m(t), x)) - f_j(u_j^*) \right] \, dx.
\]

From the boundary condition (2) and the proof of Theorem 1 in [13], we get

\[
\sum_{k=1}^{l} \int_{\Omega} (u_i - u_i^*) \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial (u_i - u_i^*)}{\partial x_k} \right) \, dx = - \sum_{k=1}^{l} \int_{\Omega} D_{ik} \left( \frac{\partial (u_i - u_i^*)}{\partial x_k} \right)^2 \, dx.
\]

From (6–8), hypothesis (H1), and Hölder integral inequality, we have

\[
\frac{d\|u_i - u_i^*\|^2}{dt} \leq -2a_i\|u_i - u_i^*\|^2 + 2 \sum_{j=1}^{n} |b_{ij}| F_i \|u_i - u_i^*\|_2 \cdot \|u_j - u_j^*\|_2 \\
+ 2 \sum_{j=1}^{n} \sum_{m=1}^{r} |c_{ij}^{(m)}| \cdot \|u_i - u_i^*\|_2 \cdot F_j \cdot \|u_j(t - \tau_m(t), x) - u_j^*\|_2
\]
**Theorem 1.** Suppose that the output functions $f_j(\cdot)$ ($j = 1, 2, \cdots, n$) satisfy the hypotheses (H1)–(H2). If there exist constants $\lambda_i > 0$ and the system parameters $a_i, b_{ij}, c_{ij}^{(m)}$ such that

$$
\begin{align*}
-2\lambda_i a_i + \lambda_i \sum_{j=1}^{n} |b_{ij}| + F_i^2 \sum_{j=1}^{n} \lambda_j |b_{ji}|
+ \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| + \frac{1}{1-d} F_i^2 \sum_{m=1}^{r} \sum_{j=1}^{n} \lambda_j |c_{ij}^{(m)}| e^{2\varepsilon \tau_i} \leq 0
\end{align*}
$$

(10)

Then the equilibrium $u^*$ of the system (1) is globally exponentially stable.

**Proof.** It is known that bounded activation functions always guarantee the existence of an equilibrium point for system (1). The uniqueness of the equilibrium point can follow from the global exponential stability to be established below.

Since there exist constants $\lambda_i > 0$ ($i = 1, 2, \cdots, n$) such that (10) hold, we can choose a small positive constant $\varepsilon > 0$, such that

$$
\begin{align*}
\lambda_i 2(\varepsilon - a_i) + \lambda_i \sum_{j=1}^{n} |b_{ij}| + F_i^2 \sum_{j=1}^{n} \lambda_j |b_{ji}|
+ \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| + \frac{1}{1-d} F_i^2 \sum_{m=1}^{r} \sum_{j=1}^{n} \lambda_j |c_{ij}^{(m)}| e^{2\varepsilon \tau_i} \leq 0
\end{align*}
$$

(11)

To establish the global exponential stability of an equilibrium point of the delayed neural network in (11), we choose a Lyapunov functional candidate for (11) as

$$
V(t) = \int_{\Omega} \sum_{i=1}^{n} \lambda_i \left( |u_i - u_i^*|^2 e^{2\varepsilon t} + \frac{1}{1-d} \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_j^2 \int_{t-\tau_m(t)}^{t} |u_j(s, x) - u_j^*|^2 e^{2\varepsilon (s+\sigma_m)} ds \right) dx
$$

(12)

By calculating the time-derivative of $V(t)$ along the solutions of the system (1), and from (12), it follows that

$$
\dot{V}(t) \leq \int_{\Omega} \sum_{i=1}^{n} \lambda_i \left( 2(u_i - u_i^*) \frac{\partial (u_i - u_i^*)}{\partial t} e^{2\varepsilon t} + 2\varepsilon e^{2\varepsilon t} (u_i - u_i^*)^2 
+ \frac{1}{1-d} \sum_{m=1}^{r} \sum_{j=1}^{n} F_j^2 |c_{ij}^{(m)}| |u_j - u_j^*|^2 e^{2\varepsilon (t+\tau_m)} 
- e^{2\varepsilon t} \sum_{m=1}^{r} \sum_{j=1}^{n} F_j^2 |c_{ij}^{(m)}| ||u_j(t - \tau_m(t), x) - u_j^*||^2 \right) dx
$$
Then we use the inequality \(2ab \leq a^2 + b^2\) to estimate the right hand side of (13), and from (11), we have that

\[
\dot{V}(t) \leq e^{2\epsilon t} \int_{\Omega} \sum_{i=1}^{n} \lambda_i \left[ 2(u_i - u_i^*) \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial(u_i - u_i^*)}{\partial x_k} \right) + 2(\epsilon - a_i)(u_i - u_i^*)^2 + \sum_{j=1}^{n} b_{ij} \cdot (u_i - u_i^*)^2 \right. \\
+ \sum_{j=1}^{n} F_j^2 b_{ij} \cdot (u_j - u_j^*)^2 + \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| \cdot (u_i - u_i^*)^2 \\
+ \sum_{m=1}^{r} \sum_{j=1}^{n} F_j^2 |c_{ij}^{(m)}| \cdot (u_j(t - \tau_m(t), x) - u_j^*)^2 \\
+ \frac{1}{1-d} \sum_{m=1}^{r} \sum_{j=1}^{n} F_j^2 |c_{ij}^{(m)}| (u_j^* - u_j^*)^2 e^{2\epsilon \sigma_m} \\
- \sum_{m=1}^{r} \sum_{j=1}^{n} F_j^2 |c_{ij}^{(m)}| (u_j(t - \tau_m(t), x) - u_j^*)^2 \left. \right] \right] \, dx \\
= e^{2\epsilon t} \int_{\Omega} \sum_{i=1}^{n} \lambda_i \left[ 2(u_i - u_i^*) \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial(u_i - u_i^*)}{\partial x_k} \right) \right] \, dx \\
+ e^{2\epsilon t} \int_{\Omega} \sum_{i=1}^{n} \left[ \lambda_i 2(\epsilon - a_i) + \lambda_i \sum_{j=1}^{n} b_{ij} + F_i^2 \lambda_j |b_{ji}| \\
+ \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}|^2 + \frac{1}{1-d} F_i^2 \sum_{m=1}^{r} \lambda_j |c_{ij}^{(m)}| e^{2\epsilon \sigma_m} \right] (u_i - u_i^*)^2 \, dx
\]
\[ e^{2 \varepsilon t} \int_{\Omega} \sum_{i=1}^{n} \left[ \lambda_i (\varepsilon - a_i) + \lambda_i \sum_{j=1}^{n} |b_{ij}| + F_i^{2} \sum_{j=1}^{n} \lambda_j |b_{ji}| \right. \\
+ \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| + \frac{1}{1 - d} \sum_{m=1}^{r} \sum_{j=1}^{n} \lambda_j |c_{ji}^{(m)}| e^{2 \varepsilon \sigma_m} \left] \right] (u_i - u_i^*)^2 dx \leq 0 \quad (14) \]

So \( V(t) \leq V(0), \forall t \geq 0. \)

Obviously, it can be obtained from (13)

\[ V(t) \geq \min_{1 \leq i \leq n} (\lambda_i) e^{2 \varepsilon t} \int_{\Omega} \sum_{i=1}^{n} |u_i - u_i^*|^2 dx = \min_{1 \leq i \leq n} (\lambda_i) e^{2 \varepsilon t} \sum_{i=1}^{n} \|u_i - u_i^*\|_2^2, \quad (15) \]

and

\[
V(0) = \int_{\Omega} \sum_{i=1}^{n} \lambda_i \left[ |u_i(0, x) - u_i^*|^2 \\
+ \frac{1}{1 - d} \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| \int_{-\tau_m(0)}^{0} F_j^{2} |u_j(s, x) - u_j^*|^2 e^{2 \varepsilon (s + \sigma_m)} ds \right] dx \\
\leq \int_{\Omega} \left[ \max_{1 \leq i \leq n} (\lambda_i) \sum_{i=1}^{n} |\phi_u(0, x) - u_i^*|^2 \\
+ \frac{1}{1 - d} e^{2 \varepsilon \sigma_m} \sum_{m=1}^{r} \sum_{j=1}^{n} F_j^{2} \sum_{j=1}^{n} |c_{ji}^{(m)}| \int_{-\sigma}^{0} |\phi_u(s, x) - u_i^*|^2 e^{2 \varepsilon s} ds \right] dx \\
\leq \left[ \max_{1 \leq i \leq n} (\lambda_i) + \frac{1}{1 - d} \sigma_m e^{2 \varepsilon \sigma_m} \max_{1 \leq i \leq n} (F_i^{2}) \sum_{m=1}^{r} \sum_{j=1}^{n} \max_{1 \leq i \leq n} (|c_{ji}^{(m)}|) \right] ||\phi_u - u^*||_2^2 \\
\leq \min_{1 \leq i \leq n} (\lambda_i) \right) (16) \]

Let

\[ M = \min_{1 \leq i \leq n} (\lambda_i) + \frac{1}{1 - d} \sigma_m e^{2 \varepsilon \sigma_m} \max_{1 \leq i \leq n} (F_i^{2}) \sum_{m=1}^{r} \sum_{j=1}^{n} \max_{1 \leq i \leq n} (|c_{ji}^{(m)}|) \left/ \min_{1 \leq i \leq n} (\lambda_i) \right. \]

then \( M \geq 1 > 0 \) and

\[ \sum_{i=1}^{n} \|u_i - u_i^*\|_2^2 \leq M e^{-2 \varepsilon t} ||\phi_u - u^*||_2^2 \quad (17) \]

This implies that the equilibrium point of system (1) is globally exponentially stable. The proof is completed.
Remark 1. If let $b_{ij} \equiv 0$ and reduce multiple time-varying delays to constant delays, i.e., $m = 1$, we can easily obtain the conditions of Theorem 1 in [15].

Corollary 1. Suppose that the output functions $f_j(\cdot)$ ($j = 1, 2, \cdots, n$) satisfy the hypotheses (H1)–(H2). If the system parameters $a_i, b_{ij}, c_{ij}^{(m)}$ are such that

$$-
2a_i + \sum_{j=1}^{n} |b_{ij}| F_i^{2\alpha_i} + \sum_{j=1}^{n} |b_{ji}| F_i^{2\beta_i}
$$

$$+ \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_i^{2\rho_i} + \frac{1}{1-d} \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ji}^{(m)}| F_i^{2\rho_i} < 0$$

Then the equilibrium $u^*$ of the system (1) is globally exponentially stable.

Remark 2. The assumption of reaction–diffusion terms in this paper is almost the same in references [12–15]. We can see its applied meaning in references [12–15].

In the following, by means of a different Lyapunov functional, we can obtain a new criterion that is, in general, independent of Theorem 1.

Theorem 2. Suppose that the output functions $f_j(\cdot)$ ($j = 1, 2, \cdots, n$) satisfy the hypotheses (H1)–(H2). If the system parameters $a_i, b_{ij}, c_{ij}^{(m)}$ are such that

$$-
2a_i + \sum_{j=1}^{n} |b_{ij}| F_j^{2\rho_i} + \sum_{j=1}^{n} |b_{ji}| F_j^{2\rho_i}
$$

$$+ \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_j^{2\rho_i} + \frac{1}{1-d} \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ji}^{(m)}| F_j^{2\rho_i} < 0$$

where $a_{ij} + \alpha_{ij} = 1, \beta_{ij} + \beta_{ij} = 1, \rho_{ij} + \rho_{ij} = 1, \theta_{ij} + \theta_{ij} = 1, i, j = 1, 2, \cdots, n, F_j$ are constant numbers in (H1). Then the equilibrium $u^*$ of the system (1) is globally exponentially stable.

Proof. It is known that bounded activation functions always guarantee the existence of an equilibrium point for system (1). The uniqueness of the equilibrium point can follow from the global asymptotic stability to be established below.

Since there exist constants $a_{ij}, \alpha_{ij}, \beta_{ij}, \rho_{ij}, \rho_{ij}, \theta_{ij}, \theta_{ij} (i, j = 1, 2, \cdots, n)$ such that (19) holds, we can choose a small positive constant $\varepsilon > 0$, such that

$$\left\{ (\varepsilon - a_i) + \frac{1}{2} \sum_{j=1}^{n} |b_{ij}|^{2p_{ij}} F_j^{2\theta_j} + \frac{1}{2} \sum_{j=1}^{n} |b_{ji}|^{2p_{ij}} F_j^{2\theta_j}
$$

$$+ \frac{1}{2} \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ij}^{(m)}|^{2\alpha_{ij}} F_j^{2\rho_{ij}} + \frac{1}{2(1-d)} \sum_{m=1}^{r} \sum_{j=1}^{n} |c_{ji}^{(m)}|^{2\alpha_{ij}} F_j^{2\rho_{ij}} \right\} \leq 0$$

(20)
We construct a Lyapunov functional

\[ V(t) = \sum_{i=1}^{n} \left\{ \frac{1}{2} \| u_i - u_i^* \|^2 e^{2\epsilon t} \right\} + \frac{1}{2(1-d)} \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}|^{2 \alpha_j} F_j^{2 \beta_j} \int_{t-\tau_m(t)}^{t} \| u_j(s,x) - u_j^* \|^2 e^{2\epsilon(s+\sigma_m)} ds \right\} \]  

(21)

By calculating the time-derivative of \( V(t) \) along the solutions of (7), we get

\[ \dot{V}(t) \leq \sum_{i=1}^{m} \left\{ e^{2\epsilon t} \left[ (\varepsilon - a_i) \| u_i - u_i^* \|^2 \right] + \sum_{j=1}^{n} \left( |b_{ij}|^{\rho_j} F_j^{\rho_j} \| u_i - u_i^* \|^2 \right) \right\} + \frac{1}{2(1-d)} \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}|^{2 \alpha_j} F_j^{2 \beta_j} \| u_j^* \|^2 \]

\[ = \sum_{i=1}^{m} \left\{ e^{2\epsilon t} \left[ (\varepsilon - a_i) \| u_i - u_i^* \|^2 \right] \right\} + \frac{1}{2(1-d)} \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}|^{2 \alpha_j} F_j^{2 \beta_j} \| u_j^* \|^2 \]

\[ \leq \sum_{i=1}^{m} \left\{ e^{2\epsilon t} \left[ (\varepsilon - a_i) \| u_i - u_i^* \|^2 \right] + \frac{1}{2} \sum_{j=1}^{n} |b_{ij}|^{2 \rho_j} F_j^{2 \beta_j} \| u_i - u_i^* \|^2 \right\} + \frac{1}{2} \sum_{j=1}^{n} |b_{ij}|^{2 \rho_j} F_j^{2 \beta_j} \| u_j - u_j^* \|^2 \]
\[ V(t) \leq V(0) \quad \forall t \geq 0. \]

From (21), we have

\[
V(t) \geq \frac{1}{2} e^{2\varepsilon t} \sum_{i=1}^{n} \|u_i - u_i^*\|_2^2
\]

and

\[
V(0) = \sum_{i=1}^{n} \left\{ \frac{1}{2} \|u_i(0, x) - u_i^*\|_2^2 \right. \\
+ \left. \frac{1}{2(1-d)} \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}| 2\alpha_j^{\prime} F_j^{2\beta_j^*} \int_{-\tau_m(0)}^{0} \|u_j(s, x) - u_j^*\|_2^2 e^{2\varepsilon (s + \sigma_m)} ds \right\}
\]

\[
\leq \left\{ \frac{1}{2} + \frac{1}{2(1-d)} \sigma_m e^{2\varepsilon \sigma_m} \max_{1 \leq i \leq n} (F_i^{2\beta_i^*}) \sum_{r=1}^{m} \sum_{j=1}^{n} \max_{1 \leq i \leq n} \left| c_{ij}^{(m)} \right| 2\alpha_j^{\prime} \right\} \|\phi_u - u^*\|_2^2
\]
Let
\[ M = \left\{ 1 + \frac{1}{1 - d} \sigma_m e^{2c \sigma_m} \max_{1 \leq i \leq n} \left( F_i^{2\beta_j} \right) \sum_{r=1}^{m} \sum_{j=1}^{n} \max_{1 \leq i \leq n} \left( |c_{ij}^{(m)}|^{2\alpha_j'} \right) \right\} \] (24)
then \( M \geq 1 > 0 \) and
\[ \sum_{i=1}^{n} \| u_i - u_i^* \|^2 \leq M e^{-2ct} \| \phi_u - u^* \|^2 \] (25)

This implies that the equilibrium point of system (1) is globally exponentially stable. The proof is completed.

**Remark 3.** Some famous neural networks models became a special case of system (1). For example, when \( D_{ik}(t, x, u) = 0 \) \( (i = 1, 2, \cdots, n) \) and \( r = 1 \), system (1) becomes a normal neural network, which has been studied in references [20–24].

**Remark 4.** It can be seen that if we choose \( \alpha_{ij} = \alpha'_{ij} = \frac{1}{2}, \beta_j = \varrho_j = 0, \rho_{ij} = \rho'_{ij} = \frac{1}{2}, \beta'_{j} = \varrho'_j = 2 \), then the condition in Theorem 2 becomes the condition in Corollary 1. Similarly, choosing the parameters properly, one can derive the following corollaries.

**Corollary 2.** Suppose that the output functions \( f_j(\cdot) \) \( (j = 1, 2, \cdots, n) \) satisfy the hypotheses (H1)–(H2). If the system parameters \( a_i, b_{ij}, c_{ij}^{(m)} \) are such that
\[ -2a_i + \sum_{j=1}^{n} |b_{ij}| F_j + \sum_{j=1}^{n} |b_{ji}| F_i + \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_j + \frac{1}{1 - d} \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_i < 0 \] (26)
where \( F_j \) are constant numbers in (H1), then the equilibrium \( u^* \) of the system (1) is globally exponentially stable.

**Corollary 3.** Suppose that the output functions \( f_j(\cdot) \) \( (j = 1, 2, \cdots, n) \) satisfy the hypotheses (H1)–(H2). If the system parameters \( a_i, b_{ij}, c_{ij}^{(m)} \) are such that
\[ -2a_i + \sum_{j=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 + \sum_{j=1}^{n} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_j + \frac{1}{1 - d} \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_i < 0 \] (27)
where \( F_j \) are constant numbers in (H1), then the equilibrium \( u^* \) of the system (1) is globally exponentially stable.
Corollary 4. Suppose that the output functions \( f_j(\cdot) \) \((j = 1, 2, \cdots, n)\) satisfy the hypotheses \((H1)-(H2)\). If the system parameters \(a_i, b_{ij}, c_{ij}^{(m)}\) are such that

\[
-2a_1 + \sum_{j=1}^{n} |b_{1j}| + \sum_{j=1}^{n} |b_{2j}| F_1^2 + \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}| F_2^2 + \frac{1}{1 - d} \sum_{r=1}^{m} \sum_{j=1}^{n} |c_{ij}^{(m)}| < 0
\]

where \(F_j\) are constant numbers in \((H1)\), then the equilibrium \(u^*\) of the system \((1)\) is globally exponentially stable.

Remark 5. As we can see, \(\lambda_i \ (i = 1, 2, \cdots, n)\) are adjustable parameters in Theorem 1, while in Theorem 2, the adjustable parameters are \(\alpha_{ij}, \alpha'_{ij}, \beta_j, \beta'_j, \rho_{ij}, \rho'_{ij}, \theta_j, \theta'_j \ (i, j = 1, 2, \cdots, n)\). Thus they provide different ways to guarantee stability when they are applied in practical application. However, if \(\lambda_i \equiv 1 \ (i = 1, 2, \cdots, n)\), Theorem 1 is deduced to a special case of Theorem 2.

4. AN ILLUSTRATIVE EXAMPLE

In this section, we give an illustrative example for our results.

Example 1. Consider a delayed neural network in \((1)\) with parameters as

\[
\begin{align*}
r & = 2, \quad a_1 = 1.7, \quad a_2 = 1.2, \quad b_{11} = -0.34, \quad b_{12} = -0.43, \quad b_{21} = 0.35, \quad b_{22} = -0.03, \\
c_{11}^{(1)} & = 0.21, \quad c_{12}^{(1)} = 0.30, \quad c_{21}^{(1)} = 0.45, \quad c_{22}^{(1)} = -0.40, \quad c_{11}^{(2)} = c_{12}^{(2)} = c_{21}^{(2)} = c_{22}^{(2)} = 0
\end{align*}
\]

the activation function is described by \(f_j(v) = \tanh(v)(j = 1, 2)\). Clearly, \(f_j(v)\) satisfies condition \((H1)-(H2)\) above, with \(F_1 = F_2 = 1\). In this example, we assume that the time delay satisfies \(0 \leq \tau(t) \leq 0.5\). Furthermore, we assume

\[
\sigma_1 = 0.5, \quad \sigma_2 = 0.4
\]

Now, it is easy to check that:

\[
\begin{align*}
\left[ -2a_1 + \sum_{j=1}^{2} |b_{1j}| + F_1^2 \sum_{j=1}^{2} |b_{2j}| + \sum_{m=1}^{2} \sum_{j=1}^{2} |c_{ij}^{(m)}| + \frac{1}{1 - 0.5} F_2^2 \sum_{m=1}^{2} \sum_{j=1}^{2} |c_{ij}^{(m)}| \right] & = -0.11 < 0
\end{align*}
\]
\[
-2a_2 + \sum_{j=1}^{2} |b_{2j}| + F_2^2 \sum_{j=1}^{2} |b_{2j}|
+ \sum_{m=1}^{2} \sum_{j=1}^{2} |c_{2j}^{(m)}| + \frac{1}{1 - 0.5} F_2^2 \sum_{m=1}^{2} \sum_{j=1}^{2} |c_{2j}^{(m)}| = -0.01 < 0 \tag{30}
\]

The conditions in Corollary 1 are satisfied, so it is easy to check the equilibrium point for the system (1) with multiple time-varying delays and reaction–diffusion terms is the global exponential stability.

5. CONCLUSION

In this paper, some criteria for the global exponential stability of recurrent neural networks with multiple time-varying delays and reaction–diffusion terms based on suitable Lyapunov functional and analytic technique are presented. The results extend and improve the earlier publications for the cases with and without reaction–diffusion terms. It is believed that these results are significant and useful for the design and applications of global exponential stability for recurrent neural networks with multiple time-varying delays and reaction–diffusion terms.

REFERENCES