ON $e$-OPEN SETS, $\mathcal{DP}^*_\mathcal{E}$-SETS AND $\mathcal{DPE}^*$-SETS AND DECOMPOSITIONS OF CONTINUITY

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الخلاصة

إنهدف الرئيسي من هذا البحث هو دراسة أنواع جديدة من صفوف المجموعات والحصول على تحليل جديد للاستمرارية. وتحقق هذا الهدف سوف تعرض ترميزات مجموعات $e$-المفتوحة $\mathcal{DP}^*_\mathcal{E}$، $\mathcal{DPE}^*$-مجموعات $\mathcal{DPE}^*$، $\mathcal{DPE}^*$-مجموعات $\mathcal{DPE}^*$، $\mathcal{DPE}^*$-مجموعات $\mathcal{DPE}^*$. كما سنقوم بدراسة خواص هذه المجموعات والعلاقة فيما بينها وبين المفاهيم المرتبطة بها. وقد حصلنا على تحليل الاستمرارية.

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The main goal of this paper is to study some new classes of sets and to obtain some new decompositions of continuity. For this aim, the notions of $e$-open sets, $DP$-sets, $DPE$-sets, $DP^*$-sets, $DPE^*$-sets, $e$-continuous functions, $DP^*$-continuous functions, and $DPE^*$-continuous functions are introduced. Properties of $e$-open sets, $DP$-sets, $DPE$-sets, $DP^*$-sets, $DPE^*$-sets and the relationships between these sets and the related concepts are investigated. Finally, some new decompositions of continuity are obtained.

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ON $\varepsilon$-OPEN SETS, $\mathcal{DP}^*$-SETS AND $\mathcal{DP}\mathcal{E}^*$-SETS AND DECOMPOSITIONS OF CONTINUITY

1. INTRODUCTION

The importance of continuity and generalized continuity is significant in various areas of mathematics and related sciences. One of them, which has been in recent years of interest to general topologists, is its decomposition. The decomposition of continuity has been studied by many authors. In those papers, the authors introduced some new classes of sets and using them obtained new decompositions of continuity. For example, Tong [1] proved that a function is continuous if and only if it is $\alpha$-continuous and $\mathcal{A}$-continuous, and Dontchev and Przemski [2] gave various other decompositions of continuity. The purpose of this note is to present some new decompositions of continuity. In the second section, the notion of $\varepsilon$-open sets which is a generalization of $\delta$-semiopen sets and $\delta$-preopen sets is introduced. Properties and the relationships of $\varepsilon$-open sets are investigated. In the third section, four new classes of sets namely $\mathcal{DP}$-sets, $\mathcal{DP}\mathcal{E}$-sets, $\mathcal{DP}^*$-sets, and $\mathcal{DP}\mathcal{E}^*$-sets are introduced and studied. In the last section, we obtain some new decompositions of continuity by using the notions of $\varepsilon$-continuous functions, $\mathcal{DP}^*$-continuous functions, and $\mathcal{DP}\mathcal{E}^*$-continuous functions.

In this paper $(X, \tau)$ and $(Y, \sigma)$ represent topological spaces. For a subset $H$ of a space $X$, $cl(H)$ and $int(H)$ denote the closure of $H$ and the interior of $H$, respectively.

A subset $H$ of a space $(X, \tau)$ is called regular open (resp. regular closed) [3] if $H = int(cl(H))$ (resp. $H = cl(int(H))$). A subset $H$ is said to be $\delta$-open [4] if for each $x \in H$ there exists a regular open set $G$ such that $x \in G \subset H$. A point $x \in X$ is called a $\delta$-cluster point of $H$ [4] if $H \cap int(cl(U)) \neq \emptyset$ for each open set $U$ containing $x$. The set of all $\delta$-cluster points of $H$ is called the $\delta$-closure of $H$ and is denoted by $\delta-cl(H)$. If $\delta-cl(H) = H$, then $H$ is said to be $\delta$-closed. The set $\{x \in X : x \in U \subset H$ for some regular open set $U$ of $X\}$ is called the $\delta$-interior of $H$ and is denoted by $\delta-int(H)$.

A subset $H$ of a space $(X, \tau)$ is called semiopen [5] (resp. $\alpha$-open [6], $\beta$-open [7], $b$-open [8], or $sp$-open [2], preopen [9], $\delta$-preopen [10], $\delta$-semiopen [11]) if $H \subset cl(int(H))$ (resp. $H \subset int(cl(H))), H \subset cl(int(H))), H \subset cl(int(H))), H \subset cl(int(H)))$. The complement of a $\delta$-semiopen set (resp. a $\delta$-preopen set) is called $\delta$-semiclosed (resp. $\delta$-preclosed). The intersection of all $\delta$-semiclosed (resp. $\delta$-preclosed) sets containing a set $H$ in a topological space $X$ is called the $\delta$-semiclosure [11] (resp. $\delta$-preclosure [10]) of $H$ and it is denoted by $\delta-scl(H)$ (resp. $\delta-pcl(H)$). The union of all $\delta$-semiopen (resp. $\delta$-preopen) sets contained in a set $H$ in a topological space $X$ is called the $\delta$-semiinterior [11] (resp. $\delta$-preinterior [10]) of $H$ and it is denoted by $\delta-sint(H)$ (resp. $\delta-pint(H)$).

Definition 1.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\alpha$-continuous [12] (resp. precontinuous [9], $\delta$-almost continuous [10], $\delta$-semicontinuous [13]) if $f^{-1}(V)$ is $\alpha$-open (resp. preopen, $\delta$-preopen, $\delta$-semiopen) for each $V \in \sigma$.

Lemma 1.2. (11)) The following hold for a subset $H$ of a space $X$:

1. $\delta-pcl(H) = H \cup cl(\delta-int(H))$ and $\delta-pint(H) = H \cap int(\delta-cl(H))$;
2. $\delta-pcl(\delta-pint(H)) = \delta-pint(H) \cup cl(\delta-int(H))$ and $\delta-pint(\delta-pcl(H)) = \delta-pcl(H) \cap int(\delta-cl(H))$;
3. $\delta-sint(H) = H \cap cl(\delta-int(H))$ and $\delta-scl(H) = H \cup int(\delta-cl(H))$;

\( (4) \delta \text{-int}(\delta \text{-scl}(H)) = \text{int}(\delta \text{-cl}(H)) \) and \( \delta \text{-int}(\delta \text{-pcl}(H)) = \text{int}(\delta \text{-cl}(H)) \);

\( (5) \delta \text{-pcl}(\delta \text{-sint}(H)) = \text{cl}(\delta \text{-int}(H)) \) and \( \delta \text{-scl}(\delta \text{-pint}(H)) = \text{int}(\delta \text{-cl}(H)) \);

\( (6) \delta \text{-scl}(\delta \text{-sint}(H)) = \delta \text{-sint}(H) \cup \text{int}(\delta \text{-cl}(H)) \).

2. \( \varepsilon \)-OPEN SETS IN TOPOLOGICAL SPACES

**Definition 2.1.** A subset \( H \) of a space \( X \) is called:

1. \( \varepsilon \)-Open if \( H \subset \text{cl}(\delta \text{-int}(H)) \cup \text{int}(\delta \text{-cl}(H)) \);

2. \( \varepsilon \)-Closed if \( \text{cl}(\delta \text{-int}(H)) \cap \text{int}(\delta \text{-cl}(H)) \subset H \).

**Remark 2.2.** The following diagram holds for a subset \( H \) of a space \( X \):

\[
\begin{array}{ccccccc}
\text{open} & \to & \alpha \text{-open} & \to & \text{preopen} & \to & b \text{-open} & \to & \beta \text{-open} \\
\uparrow & & & & & & \downarrow & & \\
\delta \text{-open} & \searrow & \delta \text{-preopen} & & & & & \\
& & & \downarrow & & & & \\
\delta \text{-semiopen} & \to & \varepsilon \text{-open} & & & & & \\
\end{array}
\]

None of these implications is reversible as shown in the following examples:

**Example 2.3.** Let \( X = \{a, b, c\} \) and let \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then the set \( \{b, c\} \) is \( \varepsilon \)-open but it is not \( \delta \)-preopen.

**Example 2.4.** Let \( X = \{a, b, c\} \) and let \( \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\} \). Then the set \( \{a, c\} \) is \( \varepsilon \)-open but it is not \( \delta \)-semiopen.

Examples for the other implications are shown in the related papers.

**Remark 2.5.** The following example shows that the notions of \( \varepsilon \)-open set and \( b \)-open set and the notions of \( \varepsilon \)-open set and \( \beta \)-open set and the notions of \( \varepsilon \)-open set and semiopen set are independent:

**Example 2.6.** Let \( X = \{x, y, w, z\} \) and let \( \tau = \{\emptyset, \{x\}, \{w\}, \{x, y\}, \{x, w\}, \{x, y, w\}, \{x, w, z\}, X\} \). Then:

1. the set \( \{y, w\} \) is \( \varepsilon \)-open but it is not \( \beta \)-open and so it is neither semiopen nor \( b \)-open;

2. the set \( \{x, z\} \) is not \( \varepsilon \)-open but it is semiopen and so it is both \( b \)-open and \( \beta \)-open.

**Theorem 2.7.** Let \( X \) be a topological space and \( H \subset X \). Then \( H \) is \( \varepsilon \)-open if and only if \( H = \delta \text{-pint}(H) \cup \delta \text{-sint}(H) \).
Proof. Let $H$ be $e$-open. Then $H \subset cl(\delta-int(H)) \cup int(\delta-cl(H))$. By Lemma 1.2, we have

$$
\delta\text{-pint}(H) \cup \delta\text{-sint}(H) = [H \cap int(\delta-cl(H))] \cup [H \cap cl(\delta-int(H))] \\
= H \cap [int(\delta-cl(H)) \cup cl(\delta-int(H))] \\
= H 
$$

Conversely, suppose $H = \delta\text{-pint}(H) \cup \delta\text{-sint}(H)$. By Lemma 1.2, we have

$$
H = \delta\text{-pint}(H) \cup \delta\text{-sint}(H) \\
= [H \cap int(\delta-cl(H))] \cup [H \cap cl(\delta-int(H))] \\
\subset int(\delta-cl(H)) \cup cl(\delta-int(H)) 
$$

Thus, $H$ is $e$-open. □

**Theorem 2.8.** We have the following properties:

(1) The union of any family of $e$-open sets is an $e$-open set;

(2) The intersection of any family of $e$-closed sets is an $e$-closed set.

**Definition 2.9.** Let $H$ be a subset of a space $X$. The intersection of all $e$-closed sets containing $H$ is called the $e$-closure of $H$ and is denoted by $e-cl(H)$.

**Theorem 2.10.** The following holds for a subset $H$ of a space $X$:

$$
e-cl(H) = \delta\text{-pcl}(H) \cap \delta\text{-scl}(H)
$$

Proof. It is obvious that we have always $e-cl(H) \subset \delta\text{-pcl}(H) \cap \delta\text{-scl}(H)$.

Conversely, we have

$$
e-cl(H) \supset cl(\delta\text{-int}(e-cl(H))) \cap int(\delta\text{-cl}(e-cl(H))) \\
\supset cl(\delta\text{-int}(H)) \cap int(\delta\text{-cl}(H))
$$

since $e-cl(H)$ is $e$-closed. Hence, by Lemma 1.2,

$$
\delta\text{-pcl}(H) \cap \delta\text{-scl}(H) = [H \cup cl(\delta\text{-int}(H))] \cap [H \cup int(\delta\text{-cl}(H))] \subset e-cl(H) \quad \square
$$

**Definition 2.11.** Let $H$ be a subset of a space $X$. The $e$-interior of $H$, denoted by $e\text{-int}(H)$, is defined by the union of all $e$-open sets contained in $H$. 

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Theorem 2.12. The following holds for a subset $H$ of a space $X$:

$$e\text{-}int(H) = \delta\text{-}pint(H) \cup \delta\text{-}sint(H)$$

Proof. This follows from Theorem 2.10. \qed

Theorem 2.13. Let $V$ be a subset of a topological space $X$. Then the following hold:

1. $V$ is $\delta$-preopen if and only if $V \subset \delta\text{-}pint(\delta\text{-}pcl(V))$;

2. $V$ is $e$-open if and only if $V \subset \delta\text{-}pcl(\delta\text{-}pint(V))$.

Proof.

(1): Let $V$ be $\delta$-preopen. Then $\delta\text{-}pint(V) = V$ and also $V \subset \delta\text{-}pint(\delta\text{-}pcl(V))$.

Conversely, let $V \subset \delta\text{-}pint(\delta\text{-}pcl(V))$. By Lemma 1.2, we have

$$V \subset \delta\text{-}pint(\delta\text{-}pcl(V)) \subset \delta\text{-}pint(\delta\text{-}cl(V))$$

$$= \delta\text{-}cl(V) \cap int(\delta\text{-}cl(V))$$

$$= int(\delta\text{-}cl(V))$$

Hence, $V$ is $\delta$-preopen.

(2): Let $V$ be $e$-open. Then $V \subset cl(\delta\text{-}int(V)) \cup int(\delta\text{-}cl(V))$. By Lemma 1.2, we have

$$V \subset [cl(\delta\text{-}int(V)) \cup int(\delta\text{-}cl(V))] \cap V = [cl(\delta\text{-}int(V)) \cap V] \cup [int(\delta\text{-}cl(V)) \cap V]$$

$$\subset \delta\text{-}pint(V) \cup cl(\delta\text{-}int(V))$$

$$= \delta\text{-}pcl(\delta\text{-}pint(V))$$

Conversely, suppose $V \subset \delta\text{-}pcl(\delta\text{-}pint(V))$. By Lemma 1.2, we have

$$V \subset \delta\text{-}pcl(\delta\text{-}pint(V)) = \delta\text{-}pint(V) \cup cl(\delta\text{-}int(V))$$

$$= [V \cap int(\delta\text{-}cl(V))] \cup cl(\delta\text{-}int(V))$$

$$\subset int(\delta\text{-}cl(V)) \cup cl(\delta\text{-}int(V))$$

Hence, $V$ is $e$-open. \qed

Lemma 2.14. The following hold for a subset $H$ of a space $X$:

1. $cl(\delta\text{-}int(H)) = \delta\text{-}cl(\delta\text{-}int(H))$;

2. $int(\delta\text{-}cl(H)) = \delta\text{-}int(\delta\text{-}cl(H))$. 


Theorem 2.15. The following hold for a subset $H$ of a space $X$:

(1) $e\text{-cl}(\delta\text{-int}(H)) = \text{int}(\text{cl}(\delta\text{-int}(H)))$;

(2) $\delta\text{-int}(e\text{-cl}(H)) = \text{int}(\text{cl}(\delta\text{-int}(H)))$;

(3) $e\text{-int}(\delta\text{-cl}(H)) = \delta\text{-cl}(e\text{-int}(H)) = \text{cl}(\text{int}(\delta\text{-cl}(H)))$;

(4) $e\text{-cl}(\delta\text{-sint}(H)) = \delta\text{-scl}(\delta\text{-sint}(H))$;

(5) $\delta\text{-pint}(e\text{-cl}(H)) = \delta\text{-pint}(\delta\text{-pcl}(H))$;

(6) $\delta\text{-sint}(e\text{-cl}(H)) = \text{cl}(\delta\text{-int}(H)) \cap \delta\text{-scl}(H)$;

(7) $e\text{-int}(\delta\text{-scl}(H)) = \delta\text{-sint}(\delta\text{-scl}(H))$;

(8) $\delta\text{-pcl}(e\text{-int}(H)) = \delta\text{-pcl}(\delta\text{-pint}(H))$;

(9) $\delta\text{-scl}(e\text{-int}(H)) = \text{int}(\delta\text{-cl}(H)) \cup \delta\text{-sint}(H)$.

Proof.

(1) : By Lemma 1.2 and Theorem 2.10,

\[
e\text{-cl}(\delta\text{-int}(H)) = \delta\text{-pcl}(\delta\text{-int}(H)) \cap \delta\text{-scl}(\delta\text{-int}(H))
\]

\[
= [\delta\text{-int}(H) \cup \text{cl}(\delta\text{-int}(\delta\text{-int}(H)))] \cap [\delta\text{-int}(H) \cup \text{int}(\delta\text{-cl}(\delta\text{-int}(H)))]
\]

\[
= \delta\text{-int}(H) \cup [\text{cl}(\delta\text{-int}(H)) \cap \text{int}(\delta\text{-cl}(\delta\text{-int}(H)))]
\]

\[
= \delta\text{-int}(H) \cup [\text{cl}(\delta\text{-int}(H)) \cap \text{int}(\text{cl}(\delta\text{-int}(H)))]
\]

\[
= \delta\text{-int}(H) \cup \text{int}(\text{cl}(\delta\text{-int}(H)))
\]

\[
= \text{int}(\text{cl}(\delta\text{-int}(H)))
\]

(2) : By Lemma 1.2 and Theorem 2.10, we obtain

\[
\delta\text{-int}(e\text{-cl}(H)) = \delta\text{-int}(\delta\text{-pcl}(H) \cap \delta\text{-scl}(H))
\]

\[
= \delta\text{-int}(\delta\text{-pcl}(H)) \cap \delta\text{-int}(\delta\text{-scl}(H))
\]

\[
= \text{int}(\text{cl}(\delta\text{-int}(H))) \cap \text{int}(\delta\text{-cl}(H))
\]

\[
= \text{int}(\text{cl}(\delta\text{-int}(H)))
\]

(3) : Follows from (1) and (2).
(4) : By Lemma 1.2 and Theorem 2.10,

\[ e-cl(\delta-sint(H)) = \delta-pcl(\delta-sint(H)) \cap \delta-scl(\delta-sint(H)) \]
\[ = cl(\delta-int(H)) \cap [\delta-sint(H) \cup int(cl(\delta-int(H)))] \]
\[ = \delta-sint(H) \cup int(cl(\delta-int(H))) \]
\[ = \delta-scl(\delta-sint(H)) \]

(5) : By Theorem 2.10, we always have \( \delta-pint(e-cl(H)) \subset \delta-pint(\delta-pcl(H)) \).

Conversely, by Lemma 1.2 and Theorem 2.10, we obtain

\[ \delta-pint(e-cl(H)) = \delta-pint(\delta-pcl(H) \cap \delta-scl(H)) \]
\[ = \delta-pcl(H) \cap \delta-scl(H) \cap int(\delta-cl(\delta-pcl(H) \cap \delta-scl(H))) \]
\[ \subset \delta-pcl(H) \cap int(\delta-cl(H)) \cap \delta-scl(H) \cap int(\delta-cl(\delta-pcl(H) \cap \delta-scl(H))) \]
\[ = \delta-pcl(H) \cap int(\delta-cl(H)) \cap \delta-cl(\delta-pcl(H) \cap \delta-scl(H)) \]
\[ = \delta-pcl(H) \cap int(\delta-cl(H)) \]
\[ = \delta-pint(\delta-pcl(H)) \]

Thus, \( \delta-pint(e-cl(H)) = \delta-pint(\delta-pcl(H)) \).

(6) : Let \( H \) be a subset of \( X \). By Theorem 2.10 and Lemma 1.2,

\[ \delta-sint(e-cl(H)) \subset \delta-sint(\delta-pcl(H)) \]
\[ = cl(\delta-int(H)) \]

and

\[ \delta-sint(e-cl(H)) \subset \delta-sint(\delta-scl(H)) \]
\[ \subset \delta-scl(H) \]

Thus, \( \delta-sint(e-cl(H)) \subset \delta-scl(H) \cap cl(\delta-int(H)) \).
Conversely, by Lemma 1.2 and Theorem 2.10,

\[ \delta \cdot \text{sint}(e \cdot \text{cl}(H)) = \delta \cdot \text{sint}(\delta \cdot \text{pcl}(H) \cap \delta \cdot \text{scl}(H)) \]

\[ = \delta \cdot \text{pcl}(H) \cap \delta \cdot \text{scl}(H) \cap \text{cl}(\delta \cdot \text{int}(\delta \cdot \text{pcl}(H) \cap \delta \cdot \text{scl}(H))) \]

\[ \supset \delta \cdot \text{pcl}(H) \cap \text{cl}(\delta \cdot \text{int}(H)) \cap \delta \cdot \text{scl}(H) \cap \text{cl}(\delta \cdot \text{int}(\delta \cdot \text{pcl}(H) \cap \delta \cdot \text{scl}(H))) \]

\[ = \text{cl}(\delta \cdot \text{int}(H)) \cap \delta \cdot \text{scl}(H) \cap \text{cl}(\delta \cdot \text{int}(\delta \cdot \text{pcl}(H) \cap \delta \cdot \text{scl}(H))) \]

\[ = \delta \cdot \text{scl}(H) \cap \text{cl}(\delta \cdot \text{int}(H)) \]

Hence, \( \delta \cdot \text{sint}(e \cdot \text{cl}(H)) = \delta \cdot \text{scl}(H) \cap \delta \cdot \text{int}(H) \).

(7), (8), and (9) follow from (4), (5), and (6), respectively. \( \Box \)

3. \( \mathcal{DP} \)-SETS AND \( \mathcal{DP}\mathcal{E} \)-SETS

**Definition 3.1.** A subset \( V \) of a space \( X \) is said to be:

1. a \( \mathcal{DP} \)-set if \( \delta \cdot \text{pcl}(\delta \cdot \text{pint}(V)) = \text{int}(V) \),
2. a \( \mathcal{DP}\mathcal{E} \)-set if \( \delta \cdot \text{pint}(\delta \cdot \text{pcl}(V)) = \text{int}(V) \),
3. a \( \mathcal{DP}^* \)-set if there exist, an open set \( A \) and a \( \mathcal{DP} \)-set \( B \) such that \( V = A \cap B \),
4. a \( \mathcal{DP}\mathcal{E}^* \)-set if there exist, an open set \( A \) and a \( \mathcal{DP}\mathcal{E} \)-set \( B \) such that \( V = A \cap B \).

**Theorem 3.2.** The following holds for a subset \( H \) of a space \( X \):

\[ \delta \cdot \text{pint}(\delta \cdot \text{pcl}(H)) \subset \delta \cdot \text{pcl}(\delta \cdot \text{pint}(H)) \]

**Proof.** By Lemma 1.2, we have

\[ \delta \cdot \text{pint}(\delta \cdot \text{pcl}(H)) = \delta \cdot \text{pcl}(H) \cap \text{int}(\delta \cdot \text{cl}(H)) \]

\[ = [H \cup \text{cl}(\delta \cdot \text{int}(H))] \cap \text{int}(\delta \cdot \text{cl}(H)) \]

\[ = [H \cap \text{int}(\delta \cdot \text{cl}(H))] \cup [\text{cl}(\delta \cdot \text{int}(H)) \cap \text{int}(\delta \cdot \text{cl}(H))] \]

\[ \subset [H \cap \text{int}(\delta \cdot \text{cl}(H))] \cup \text{cl}(\delta \cdot \text{int}(H)) \]

\[ = \delta \cdot \text{pint}(H) \cup \text{cl}(\delta \cdot \text{int}(H)) \]

\[ = \delta \cdot \text{pcl}(\delta \cdot \text{pint}(H)) \]

Thus, \( \delta \cdot \text{pint}(\delta \cdot \text{pcl}(H)) \subset \delta \cdot \text{pcl}(\delta \cdot \text{pint}(H)) \). \( \Box \)
Theorem 3.3. Let $V$ be a subset of a space $X$. The following hold:

1. if $V$ is a $\mathcal{DP}$-set, then it is a $\mathcal{DPE}$-set;
2. if $V$ is a $\mathcal{DPE}$-set, then it is $e$-closed;
3. if $V$ is a $\mathcal{DP}$-set, then it is $\delta$-preclosed.

Proof.

1: Let $V$ be a $\mathcal{DP}$-set. Then, by Theorem 3.2,
\[
\text{int}(V) \subset \text{int}(\delta-pcl(V)) \subset \delta-\text{pint}(\delta-pcl(V)) \\
\subset \delta-pcl(\delta-\text{pint}(V)) \\
= \text{int}(V)
\]

Thus, $\delta-\text{pint}(\delta-pcl(V)) = \text{int}(V)$ and hence $V$ is a $\mathcal{DPE}$-set.

2: Let $V$ be a $\mathcal{DPE}$-set. Then by Lemma 1.2,
\[
V \supset \text{int}(V) = \delta-\text{pint}(\delta-pcl(V)) = \delta-pcl(V) \cap \text{int}(\delta-cl(V)) \\
= (V \cup cl(\delta-int(V))) \cap \text{int}(\delta-cl(V)) \\
\supset cl(\delta-int(V)) \cap \text{int}(\delta-cl(V))
\]

Hence, $V$ is $e$-closed.

3: Let $V$ be a $\mathcal{DP}$-set. Then we obtain
\[
V \supset \text{int}(V) = \delta-pcl(\delta-\text{pint}(V)) \\
= \delta-\text{pint}(V) \cup cl(\delta-int(V)) \\
\supset cl(\delta-int(V))
\]

since $\delta-pcl(\delta-\text{pint}(V)) = \delta-\text{pint}(V) \cup cl(\delta-int(V))$. Thus, $V$ is $\delta$-preclosed. \qed

The following examples show that these implications are not reversible.

Example 3.4. Let $X = \{x, y, w, z\}$ and let $\tau = \{\emptyset, \{y\}, \{z\}, \{y, z\}, X\}$. Then the set $\{x, y, w\}$ is $\delta$-preclosed and a $\mathcal{DPE}$-set but it is not a $\mathcal{DP}$-set.

Example 3.5. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a, d\}, \{c\}, \{a, c, d\}\}$. Then the set $\{a, c\}$ is $e$-closed but it is not a $\mathcal{DPE}$-set.
Remark 3.6. We have the following properties:

1. Every $D\mathcal{P}$-set is a $D\mathcal{P}^*$-set;
2. Every $D\mathcal{P}\mathcal{E}$-set is a $D\mathcal{P}\mathcal{E}^*$-set;
3. Every $D\mathcal{P}^*$-set is a $D\mathcal{P}\mathcal{E}^*$-set;
4. Every open set is a $D\mathcal{P}^*$-set and so a $D\mathcal{P}\mathcal{E}^*$-set.

None of these implications is reversible as shown in the following examples:

Example 3.7. Let $X = \{x, y, w, z\}$ and let $\tau = \{\emptyset, \{y\}, \{z\}, \{y,z\}, X\}$. Then:

1. the set $\{x, w\}$ is a $D\mathcal{P}^*$-set but it is not open;
2. the set $\{x, y, w\}$ is a $D\mathcal{P}\mathcal{E}^*$-set but it is not a $D\mathcal{P}^*$-set.

Example 3.8. Let $X = \{x, y, w, z\}$ and let $\tau = \{\emptyset, \{x\}, \{w\}, \{x, y\}, \{x, w\}, \{x, y, w\}, \{x, w, z\}, X\}$. Then:

1. the set $\{x, w\}$ is a $D\mathcal{P}^*$-set but it is not a $D\mathcal{P}$-set;
2. the set $\{x, y, w\}$ is a $D\mathcal{P}\mathcal{E}^*$-set but it is not a $D\mathcal{P}\mathcal{E}$-set.

Theorem 3.9. For a subset $V$ of a topological space $X$ the following are equivalent:

1. $V$ is open;
2. $V$ is $\alpha$-open and a $D\mathcal{P}^*$-set;
3. $V$ is $\delta$-preopen and a $D\mathcal{P}^*$-set;
4. $V$ is $e$-open and a $D\mathcal{P}^*$-set;
5. $V$ is $\alpha$-open and a $D\mathcal{P}\mathcal{E}^*$-set;
6. $V$ is preopen and a $D\mathcal{P}\mathcal{E}^*$-set;
7. $V$ is $\delta$-preopen and a $D\mathcal{P}\mathcal{E}^*$-set.

Proof.

1. $\Rightarrow$ (2) : Obvious, since every open set is $\alpha$-open and a $D\mathcal{P}^*$-set.
2. $\Rightarrow$ (3) : Obvious.
3. $\Rightarrow$ (4) : Obvious.
4. $\Rightarrow$ (5) : Obvious, since every open set is $\alpha$-open and a $D\mathcal{P}\mathcal{E}^*$-set.
(5) ⇒ (6) : Obvious.

(6) ⇒ (7) : Obvious.

(4) ⇒ (1) : Let \( V \) be \( e \)-open and a \( \mathcal{DP}^* \)-set. Then there exist, an open set \( A \) and a \( \mathcal{DP} \)-set \( B \) such that \( V = A \cap B \). Also, we obtain

\[
V \subset \delta-pcl(\delta-pint(A \cap B)) \subset \delta-pcl(\delta-pint(A)) \cap \delta-pcl(\delta-pint(B))
\]

\[
= [\delta-pint(A) \cup cl(\delta-int(A))] \cap int(B)
\]

\[
\subset [\delta-pint(A) \cup cl(int(A))] \cap int(B)
\]

\[
= cl(A) \cap int(B)
\]

by Theorem 2.13 and Lemma 1.2. Since \( V \subset cl(A) \cap int(B) \cap A = int(B) \cap A, V = A \cap int(B) \) and hence \( V \) is open.

(7) ⇒ (1) : Let \( V \) be a \( \delta \)-preopen set and a \( \mathcal{DPE}^* \)-set. Then there exist an open set \( A \) and a \( \mathcal{DPE} \)-set \( B \) such that \( V = A \cap B \). Also, we have

\[
V \subset \delta-pint(\delta-pcl(V)) = \delta-pint(\delta-pcl(A \cap B))
\]

\[
\subset \delta-pint(\delta-pcl(A)) \cap \delta-pint(\delta-pcl(B))
\]

\[
= \delta-pint(\delta-pcl(A)) \cap int(B)
\]

\[
= (\delta-pcl(A) \cap int(\delta-cl(A))) \cap int(B)
\]

\[
\subset int(\delta-cl(A)) \cap int(B)
\]

Since \( V \subset int(\delta-cl(A)) \cap int(B) \cap A = A \cap int(B), V = A \cap int(B) \) and hence \( V \) is open. \( \Box \)

4. DECOMPOSITIONS OF CONTINUITY

**Definition 4.1.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be

1. \( e \)-continuous if \( f^{-1}(V) \) is \( e \)-open in \( X \) for every \( V \in \sigma \),
2. \( \mathcal{DP}^* \)-continuous if \( f^{-1}(V) \) is a \( \mathcal{DP}^* \)-set of \( X \) for every \( V \in \sigma \),
3. \( \mathcal{DPE}^* \)-continuous if \( f^{-1}(V) \) is a \( \mathcal{DPE}^* \)-set of \( X \) for every \( V \in \sigma \).

**Remark 4.2.** Let \( f : X \rightarrow Y \) be a function. The following hold:

1. \( \delta \)-almost continuous \( \rightarrow \) \( e \)-continuous

\[
\uparrow
\]

\( \delta \)-semicontinuous
(2) If \( f \) is continuous, then it is \( \mathcal{DP}^* \)-continuous.

(3) If \( f \) is \( \mathcal{DP}^* \)-continuous, then it is \( \mathcal{DP}\mathcal{E}^* \)-continuous.

These implications are not reversible as shown in the following examples:

**Example 4.3.** Let \( X = Y = \{x, y, w, z\} \) and \( \tau = \{\emptyset, \{y\}, \{z\}, \{y, z\}, X\} = \sigma \). Then

(1) the function \( f : (X, \tau) \to (Y, \sigma) \), defined as: \( f(x) = y, f(y) = x, f(w) = z, f(z) = x \), is \( \mathcal{DP}^* \)-continuous but it is not continuous.

(2) the function \( h : (X, \tau) \to (Y, \sigma) \), defined as: \( h(x) = y, h(y) = y, h(w) = y, h(z) = w \), is \( \mathcal{DP}\mathcal{E}^* \)-continuous but it is not \( \mathcal{DP}^* \)-continuous.

**Example 4.4.** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then the function \( f : (X, \tau) \to (Y, \tau) \), defined as: \( f(a) = b, f(b) = a, f(c) = a \), is \( e \)-continuous but it is not \( \delta \)-almost continuous.

**Example 4.5.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a, c\}, \{b, c\}\} \). Then the identity function \( i : (X, \tau) \to (X, \tau) \) is \( e \)-continuous but it is not \( \delta \)-semicontinuous.

**Theorem 4.6.** The following are equivalent for a function \( f : X \to Y \):

(1) \( f \) is continuous;

(2) \( f \) is \( \alpha \)-continuous and \( \mathcal{DP}^* \)-continuous;

(3) \( f \) is \( \delta \)-almost continuous and \( \mathcal{DP}^* \)-continuous;

(4) \( f \) is \( e \)-continuous and \( \mathcal{DP}^* \)-continuous;

(5) \( f \) is \( \alpha \)-continuous and \( \mathcal{DP}\mathcal{E}^* \)-continuous;

(6) \( f \) is precontinuous and \( \mathcal{DP}\mathcal{E}^* \)-continuous;

(7) \( f \) is \( \delta \)-almost continuous and \( \mathcal{DP}\mathcal{E}^* \)-continuous.

**Proof.** The proof follows from Theorem 3.9. \( \square \)

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**REFERENCES**


