

CROSSED PRODUCT OF GROUPS. APPLICATIONS[†]

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الخلاصة:

سنقوم بمسح لامتدادات الزمرة بالزمرة، باستخدام الجداء التقاطعي بدلاً من متسلسلات الزمر التامة. لهذه الطريقة ميزات متعددة، إحداها أن الجداء التقاطعي هو عنصر عالمي. سنعرض عدة تطبيقات جديدة، وسنثبت مبرهنة عامة من نوع Schreier ونطرح بعض المسائل التي لم تحل بعد.

ABSTRACT

We survey the extensions of a group by a group using crossed products instead of exact sequences of groups. The approach has various advantages, one of them being that the crossed product is a universal object. Several new applications are given, a general Schreier type theorem is proved and a few open problems are posed.

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1. INTRODUCTION

In 1895, in a long paper on extensions of groups, O. L. Hölder [1] launched one of the most interesting problems of algebra: *the extension problem*. Let H and G be two groups. The extension problem consists of describing and classifying all groups E containing H as a normal subgroup such that $E/H \cong G$. The meaning of describing and, especially, of classifying these structures is actually part of the problem. The extension problem has been the starting point of new subjects in mathematics such as cohomology of groups ([2]), homological algebra ([3]), crossed products of groups acting on algebras ([4]), crossed products of Hopf algebras acting on algebras ([5]), crossed products for von Neumann algebras [6], *etc.*

The first notable result regarding the extension problem was given by O. L. Hölder himself (Theorem 2.9 below), who uses generators and relations to describe all extensions of a finite cyclic group by another finite cyclic group. The second major contribution regarding this problem was given by O. Schreier in 1926: he classified all extensions in case H is Abelian and the morphisms are defined such that they stabilize H and G . The set of equivalence classes of extensions (via the equivalence relation that stabilizes H and G) is in a 1-1 correspondence with the cohomology group $H^2(G, H)$. The third and most important contribution to the extension problem was given in 1947 by S. Eilenberg and S. MacLane in two fundamental papers [7]. In general, no solution is known for the general classification problem (cf. [8, page 155]) although J. Baez recently stated in [9] that the above extensions are classified by weak 2-functors $G \rightarrow \text{AUT}(H)$.

The classification part of the extension problem requires first of all a definition of the morphisms between such objects. In other words, we have to define the category of extensions of a group H by a group G . Surprisingly, these morphisms have so far received only one definition, which is not the only one possible. More precisely: a morphism between two extensions (E, i, p) and (E', i', p') of H by G is a morphism of groups $\gamma : E \rightarrow E'$ for which the following diagram is commutative:

$$\begin{array}{ccccccccc}
 1 & \rightarrow & H & \xrightarrow{i} & E & \xrightarrow{p} & G & \longrightarrow & 1 \\
 & & \downarrow \text{Id}_H & & \downarrow \gamma & & \downarrow \text{Id}_G & & \\
 1 & \rightarrow & H & \xrightarrow{i'} & E' & \xrightarrow{p'} & G & \longrightarrow & 1
 \end{array}$$

The category $\mathcal{E}_1(H, G)$ obtained this way is a groupoid (*i.e.* any morphism is an isomorphism) and all classification results proved until now on the extension problem were related only to this category. Is this the only way to define a morphism between two extensions? Of course not, they can be defined in at least two other ways, one being the following, which is the most natural from the point of view of category theory: a morphism between two extensions (E, i, p) and (E', i', p') of H by G is a triple (u, γ, v) , where $u : H \rightarrow H$, $\gamma : E \rightarrow E'$, $v : G \rightarrow G$ are morphisms of groups such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
 1 & \rightarrow & H & \xrightarrow{i} & E & \xrightarrow{p} & G & \longrightarrow & 1 \\
 & & \downarrow u & & \downarrow \gamma & & \downarrow v & & \\
 1 & \rightarrow & H & \xrightarrow{i'} & E' & \xrightarrow{p'} & G & \longrightarrow & 1
 \end{array}$$

This way we obtain a new and very different category $\mathcal{E}_2(H, G)$, which is not a groupoid any more and is connected to $\mathcal{E}_1(H, G)$ via a faithful functor

$$F : \mathcal{E}_1(H, G) \rightarrow \mathcal{E}_2(H, G), \quad F((E, i, p)) = (E, i, p), \quad F(\gamma) = (\text{Id}_H, \gamma, \text{Id}_G)$$

In this paper we will approach the classification part of the extension problem by defining the morphisms as above. The main result (Theorem 4.8) provides this classification. In order to do this, we had to replace the extensions (E, i, p) with the equivalent concept of crossed systems (H, G, α, f) . The transition from extensions to crossed systems is quite natural if we ask ourselves the following elementary question:

Let H be a group and E be a set such that $H \subseteq E$. What are all the group structures (E, \cdot) that can be defined on the set E such that $H \trianglelefteq E$ is a normal subgroup of E ?

Let (E, \cdot) be such a group structure and $G := E/H$ be its quotient group. Then, as a set, $E \cong H \times G$ and hence the problem can be restated as follows: let H and G be two groups; what are all the group structures that can be defined on the set $H \times G$ with the property that $H \cong H \times \{1\} \trianglelefteq H \times G$? The set of these structures is in a one to one correspondence with the set of all crossed systems (H, G, α, f) , where $\alpha : G \rightarrow \text{Aut}(H)$ is a weak action and $f : G \times G \rightarrow H$ is an α -cocycle. Fixing the groups H and G and denoting by $\text{Crossed}(H, G)$ the set of all normalized crossed systems we shall identify three categories having the same class of objects, namely the set $\text{Crossed}(H, G)$. Thus, the classification part of the extension problem can be restated in a much more precise, categorical way: describe the skeleton of these categories. The main result of this paper describes the skeleton of the category $\mathcal{E}_2(H, G)$ (Theorem 4.8). The crossed product satisfies two universal properties: it is an initial object in a category but also a final object in another category which is not a dual of the first one (Theorem 3.1). This result has several applications; in particular, the set of all (iso)morphisms between two crossed products $H \#_{\alpha}^f G$ and $H \#_{\alpha'}^{f'} G$ is explicitly described. It is in a one to one correspondence with the set of all quadruples (u, r, v, s) , where $u : H \rightarrow H$, $r : G \rightarrow H$, $v : G \rightarrow G$ are maps, and $s : H \rightarrow G$ is a morphism of groups satisfying certain compatibility conditions (Corollary 3.2, Corollary 3.3). On the route other interesting results are derived and a few open questions are posed.

2. PRELIMINARIES

2.1. Definitions and Notation

Let us fix the notations that will be used throughout the paper. $|A|$ denotes the number of elements of a finite set A and C_n will be a cyclic group of order n generated by a : $C_n = \{1, a, a^2, \dots, a^{n-1}\}$. Let H and G be two groups. $\text{Aut}(H)$ denotes the group of automorphisms of a group H and $Z(H)$ the center of H . A map $f : G \times G \rightarrow H$ is called *symmetric* if $f(g_1, g_2) = f(g_2, g_1)$ for any $g_1, g_2 \in G$. For a map $\alpha : G \rightarrow \text{Aut}(H)$ we shall use the notation

$$\alpha(g)(h) = g \triangleright h$$

for all $g \in G$ and $h \in H$. As $\alpha(g) \in \text{Aut}(H)$ we have that

$$g \triangleright 1 = 1, \quad g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2), \quad g \triangleright h^{-1} = (g \triangleright h)^{-1} \tag{1}$$

for any $g \in G$ and $h, h_1, h_2 \in H$. The map α is called *trivial* if $g \triangleright h = h$ for all $g \in G$ and $h \in H$. If α is a morphism of groups we denote by $H \rtimes_{\alpha} G$ the semidirect product of H and G : $H \rtimes_{\alpha} G = H \times G$ as a set with the multiplication given by

$$(h_1, g_1) \cdot (h_2, g_2) := (h_1(g_1 \triangleright h_2), g_1 g_2)$$

for all $h_1, h_2 \in H, g_1, g_2 \in G$.

An *extension of H by G* is a triple (E, i, π) , where E is a group, $i : H \rightarrow E$ and $\pi : E \rightarrow G$ are morphisms of groups such that the sequence

$$1 \rightarrow H \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

is exact.

2.2. Crossed Product of Groups

We recall now a fundamental construction at the level of groups. It has served as a model for later generalizations at the level of, e.g., groups acting on rings [4], Hopf algebras acting on algebras [5], von Neumann algebras [6], quantum groupoids [10].

Definition 2.1. A *crossed system* of groups is a quadruple $\Gamma = (H, G, \alpha, f)$, where H and G are two groups, $\alpha : G \rightarrow \text{Aut}(H)$ and $f : G \times G \rightarrow H$ are two maps such that the following compatibility conditions hold:

$$\begin{aligned} g_1 \triangleright (g_2 \triangleright h) &= f(g_1, g_2) ((g_1 g_2) \triangleright h) f(g_1, g_2)^{-1} & (2) \\ f(g_1, g_2) f(g_1 g_2, g_3) &= (g_1 \triangleright f(g_2, g_3)) f(g_1, g_2 g_3) & (3) \end{aligned}$$

for all $g_1, g_2, g_3 \in G$ and $h \in H$. The crossed system $\Gamma = (H, G, \alpha, f)$ is called *normalized* if $f(1, 1) = 1$. The map $\alpha : G \rightarrow \text{Aut}(H)$ is called a *weak action* and $f : G \times G \rightarrow H$ is called an α -*cocycle*.

We denote by $\text{Crossed}(H, G)$ the set of all normalized crossed systems:

$$\text{Crossed}(H, G) = \{(\alpha, f) \mid (H, G, \alpha, f) \text{ is a normalized crossed system}\}$$

We note that if $\text{Im}(f) \subseteq Z(H)$ the condition (2) is equivalent to the fact that α is an action: $(g_1 g_2) \triangleright h = g_1 \triangleright (g_2 \triangleright h)$, for all $g_1, g_2 \in G$ and $h \in H$. First we give some useful formulas for a crossed system.

Lemma 2.2. *Let (H, G, α, f) be a crossed system. Then*

$$f(g, 1) = g \triangleright f(1, 1) \tag{4}$$

$$1 \triangleright h = f(1, 1) h f(1, 1)^{-1} \tag{5}$$

$$f(1, g) = f(1, 1) \tag{6}$$

for any $g \in G$ and $h \in H$. In particular, if (H, G, α, f) is a normalized crossed system then

$$f(1, g) = f(g, 1) = 1 \quad \text{and} \quad 1 \triangleright h = h \tag{7}$$

for any $g \in G$ and $h \in H$.

Proof. The condition (3) for $g_2 = g_3 = 1$ and $g_1 = g$ gives (4). Now if we set $g_1 = g_2 = 1$ in (2) and take into account that $\alpha(1)$ is surjective we obtain (5). If we set $g_1 = g_2 = 1$ and $g_3 = g$ in (3) and take into account (5) we obtain (6). \square

Let H and G be groups, $\alpha : G \rightarrow \text{Aut}(H)$ and $f : G \times G \rightarrow H$ two maps. Let $H \#_{\alpha}^f G := H \times G$ as a set with a binary operation defined by the formula:

$$(h_1, g_1) \cdot (h_2, g_2) := (h_1(g_1 \triangleright h_2) f(g_1, g_2), g_1 g_2) \tag{8}$$

for all $h_1, h_2 \in H, g_1, g_2 \in G$.

The following theorem gives the construction of the crossed product of groups. It is unfortunately difficult to refer to a place that would contain the proof of this version of theorem (see [2], [11], [12], [13], [14], [15], [8], [3]). Therefore, for purposes of convenience we present a short proof below:

Theorem 2.3. *Let H and G be groups $\alpha : G \rightarrow \text{Aut}(H)$ and $f : G \times G \rightarrow H$ two maps. The following statements are equivalent:*

- (1) *The multiplication on $H \#_{\alpha}^f G$ given by (8) is associative.*
- (2) *(H, G, α, f) is a crossed system.*

In this case $(H \#_{\alpha}^f G, \cdot)$ is a group with the unit $1_{H \#_{\alpha}^f G} = (f(1, 1)^{-1}, 1)$ called the *crossed product* of H and G associated to the crossed system (H, G, α, f) .

Proof. For $h_1, h_2, h_3 \in H$ and $g_1, g_2, g_3 \in G$ we have

$$[(h_1, g_1) \cdot (h_2, g_2)] \cdot (h_3, g_3) = (h_1(g_1 \triangleright h_2) f(g_1, g_2) ((g_1 g_2) \triangleright h_3) f(g_1 g_2, g_3), g_1 g_2 g_3)$$

and

$$(h_1, g_1) \cdot [(h_2, g_2) \cdot (h_3, g_3)] = (h_1(g_1 \triangleright h_2) (g_1 \triangleright (g_2 \triangleright h_3)) (g_1 \triangleright f(g_2, g_3)) f(g_1, g_2 g_3), g_1 g_2 g_3)$$

Hence, the multiplication given by (8) is associative if and only if

$$f(g_1, g_2) ((g_1 g_2) \triangleright h_3) f(g_1 g_2, g_3) = (g_1 \triangleright (g_2 \triangleright h_3)) (g_1 \triangleright f(g_2, g_3)) f(g_1, g_2 g_3) \tag{9}$$

for all $g_1, g_2, g_3 \in G$, and $h_3 \in H$. We shall prove now that (9) hold if and only if (2) and (3) hold.

Assume first that (2) and (3) hold. Then

$$\begin{aligned} \underline{f(g_1, g_2) ((g_1 g_2) \triangleright h_3) f(g_1 g_2, g_3)} &\stackrel{(2)}{=} (g_1 \triangleright (g_2 \triangleright h_3)) f(g_1, g_2) f(g_1 g_2, g_3) \\ &\stackrel{(3)}{=} (g_1 \triangleright (g_2 \triangleright h_3)) (g_1 \triangleright f(g_2, g_3)) f(g_1, g_2 g_3) \end{aligned}$$

i.e. (9) holds. Conversely, assume that (9) holds. Using (1) after we specialize $h_3 = 1$ in (9) we obtain (3). Now,

$$\begin{aligned} (g_1 \triangleright (g_2 \triangleright h))f(g_1, g_2) &\stackrel{(9)}{=} f(g_1, g_2)((g_1g_2) \triangleright h)f(g_1g_2, g_3) \\ &= [(g_1 \triangleright f(g_2, g_3))f(g_1, g_2g_3)]^{-1}f(g_1, g_2) \\ &\stackrel{(3)}{=} f(g_1, g_2)((g_1g_2) \triangleright h)f(g_1g_2, g_3)f(g_1g_2, g_3)^{-1} \\ &= f(g_1, g_2)^{-1}f(g_1, g_2) \\ &= f(g_1, g_2)((g_1g_2) \triangleright h) \end{aligned}$$

i.e. (2) holds; hence the first part of the theorem is proved.

We assume now that (H, G, α, f) is a crossed system and we prove that $(H\#_\alpha^f G, \cdot)$ is a group. For $h \in H$ and $g \in G$ we have

$$\begin{aligned} (h, g) \cdot (f(1, 1)^{-1}, 1) &= (h(g \triangleright (f(1, 1)^{-1}))f(g, 1), g) \\ &\stackrel{(4)}{=} (h(g \triangleright (f(1, 1)^{-1})(g \triangleright f(1, 1))), g) \\ &\stackrel{(1)}{=} (h(g \triangleright (f(1, 1)^{-1}f(1, 1))), g) \\ &= (h(g \triangleright 1), g) = (h, g) \end{aligned}$$

and

$$\begin{aligned} (f(1, 1)^{-1}, 1) \cdot (h, g) &= (f(1, 1)^{-1}(1 \triangleright h)f(1, g), g) \\ &\stackrel{(5)}{=} (f(1, 1)^{-1}f(1, 1)hf(1, 1)^{-1}f(1, g), g) \\ &\stackrel{(6)}{=} (hf(1, 1)^{-1}f(1, 1), g) = (h, g) \end{aligned}$$

i.e. $(f(1, 1)^{-1}, 1)$ is the unit of $(H\#_\alpha^f G, \cdot)$. Let now $(h, g) \in H\#_\alpha^f G$. Then it is easy to see that

$$(h, g)^{-1} = (f(1, 1)^{-1}f(g^{-1}, g)^{-1}(g^{-1} \triangleright h^{-1}), g^{-1})$$

is a left inverse of (h, g) . Thus $H\#_\alpha^f G$ is a monoid and any element of it has a left inverse. Then $H\#_\alpha^f G$ is a group and we are done. \square

Let $\Gamma = (H, G, \alpha, f)$ be a normalized crossed system. Then in the crossed product $H\#_\alpha^f G$ we have:

$$(h, 1) \cdot (1, g) = (h, g) \tag{10}$$

for any $h \in H$ and $g \in G$. Thus $(H \times \{1\}) \cup (\{1\} \times G)$ is a set of generators of the group $H\#_\alpha^f G$. An extension of H by G is associated to any crossed system as follows:

Corollary 2.4. *Let (H, G, α, f) be a crossed system. Then*

$$1 \rightarrow H \xrightarrow{i_H} H\#_\alpha^f G \xrightarrow{\pi_G} G \longrightarrow 1 \tag{11}$$

where $i_H(h) := (hf(1, 1)^{-1}, 1)$ and $\pi_G(h, g) := g$ for all $h \in H$ and $g \in G$ is an exact sequence of groups, i.e. $(H\#_\alpha^f G, i_H, \pi_G)$ is an extension of H by G .

Examples 2.5. (1) Let H and G be two groups and α, f be the trivial maps: i.e. $\alpha(g)(h) = h$ and $f(g_1, g_2) = 1$ for all $g, g_1, g_2 \in G$ and $h \in H$. Then $\Gamma = (H, G, \alpha, f)$ is a crossed system called the *trivial crossed system*. The crossed product $H\#_\alpha^f G = H \times G$, the direct product of H and G .

(2) Let H and G be two groups and $f : G \times G \rightarrow H$ the trivial map. Then (H, G, α, f) is a crossed system if and only if $\alpha : G \rightarrow \text{Aut}(H)$ is a morphism of groups. In this case the crossed product $H\#_\alpha^f G = H \rtimes_\alpha G$, the semidirect product of H and G .

(3) Let H and G be two groups and $\alpha : G \rightarrow \text{Aut}(H)$ the trivial action. Then (H, G, α, f) is a crossed system if and only if $\text{Im}(f) \subseteq Z(H)$ and

$$f(g_1, g_2)f(g_1g_2, g_3) = f(g_2, g_3)f(g_1, g_2g_3) \tag{12}$$

for all $g_1, g_2, g_3 \in G$, that is $f : G \times G \rightarrow Z(H)$ is a 2-cocycle as they appear in the Abelian cohomology of groups ([2], [8], [3]). The crossed product $H \#_\alpha^f G$ associated to this crossed system will be denoted by $H \times^f G$ and we shall call it the *twisted product*[†] of H and G associated to the 2-cocycle $f : G \times G \rightarrow Z(H)$. Explicitly, the multiplication of a twisted product of groups $H \times^f G$ is given by the formula:

$$(h_1, g_1) \cdot (h_2, g_2) := (h_1 h_2 f(g_1, g_2), g_1 g_2) \tag{13}$$

for all $h_1, h_2 \in H, g_1, g_2 \in G$.

The next theorem shows that any extension (E, i, π) of H by G is equivalent to a crossed product extension $(H \#_\alpha^f G, i_H, \pi_G)$. It can be also viewed as a reconstruction theorem of a group from a normal subgroup and the quotient.

Theorem 2.6. *Let (E, i, π) be an extension of H by G . Then there exists (H, G, α, f) a normalized crossed system and an isomorphism of groups $\theta : H \#_\alpha^f G \rightarrow E$ such that the following diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & H & \xrightarrow{i_H} & H \#_\alpha^f G & \xrightarrow{\pi_G} & G \longrightarrow 1 \\ & & \downarrow Id_H & & \downarrow \theta & & \downarrow Id_G \\ 1 & \longrightarrow & H & \xrightarrow{i} & E & \xrightarrow{\pi} & G \longrightarrow 1 \end{array}$$

is commutative.

Proof. For full details we refer to [8], [3]. We shall identify $H \cong i(H) \trianglelefteq E$. The crossed system is constructed as follows: let $s : G \rightarrow E$ be a section of $\pi : E \rightarrow G$ such that $s(1) = 1$ and define α and f by the formulas:

$$\alpha : G \rightarrow \text{Aut}(H), \quad \alpha(g)(h) := s(g)h s(g)^{-1} \tag{14}$$

$$f : G \times G \rightarrow H, \quad f(g_1, g_2) := s(g_1)s(g_2)s(g_1 g_2)^{-1} \tag{15}$$

for all $g, g_1, g_2 \in G$ and $h \in H$. Then (H, G, α, f) is a normalized crossed system and

$$\theta : H \#_\alpha^f G \rightarrow E, \quad \theta(h, g) := i(h)s(g)$$

is an isomorphism of groups and the diagram is commutative: $\pi(\theta(h, g)) = \pi(i(h))\pi(s(g)) = g = Id_G(\pi_G(h, g))$, for all $h \in H$ and $g \in G$. □

The next corollary shows that any crossed product of groups is isomorphic to a crossed product of a normalized crossed system.

Corollary 2.7. *Let (H, G, α, f) be a crossed system. Then there exists (H, G, α', f') a normalized crossed system such that $H \#_\alpha^f G \cong H \#_{\alpha'}^{f'} G$ (isomorphism of groups).*

Proof. It follows from the exact sequence (11) that $H \cong H \times \{1\} \trianglelefteq H \#_\alpha^f G$ and $H \#_\alpha^f G / H \times \{1\} \cong G$. Using Theorem 2.6 we obtain a normalized crossed system (H, G, α', f') such that $H \#_\alpha^f G \cong H \#_{\alpha'}^{f'} G$. □

The following result is a better formulation of the Schreier theorem [14, Theorem 12.4].

Corollary 2.8. *Let H and G be two groups. The existence of the following data is equivalent:*

- (1) *An extension of H by G .*
- (2) *A normalized crossed system (H, G, α, f) .*
- (3) *A crossed system (H, G, α, f) .*

Proof. It follows from Corollary 2.4, Theorem 2.6, and Corollary 2.7. □

[†]We borrowed the terminology from groups acting on k -algebras

Thus the extension problem of Hölder can be restated in a computational manner as follows:

Problem 1. Let H and G be two fixed groups. Describe all normalized crossed systems (H, G, α, f) and classify up to isomorphism all crossed products $H \#_{\alpha}^f G$.

The description of all extensions of a group by a group (or, equivalently in view of the corollary above, of all normalized crossed systems that can be constructed for two fixed groups) has been a central problem of group theory during the last century (see for example [2], [12], [15]). For the second part of the problem 1 (namely the classification) no solution is known in general. The first important result in the literature for the first part of problem 1 was proved by Hölder himself [14, Theorem 12.9]. It describes the crossed product of two finite cyclic groups (we refer to [16] for related results):

Theorem 2.9. A finite group E is isomorphic to a crossed product $C_n \#_{\alpha}^f C_m$ if and only if E is the group generated by two generators a and b subject to the relations

$$a^n = 1, \quad b^m = a^i, \quad b^{-1}ab = a^j$$

where $i, j \in \{0, 1, \dots, n-1\}$ such that

$$i(j-1) \equiv 0 \pmod{n}, \quad j^m \equiv 1 \pmod{n}$$

The following argument indicates the crucial importance of the previous problem: we shall prove that any finite group is isomorphic to a finite product of normalized crossed products of finite simple groups. Thus we can survey all finite groups as if we are able to compute various crossed systems starting with finite simple groups or crossed products of them. The first part of the following theorem can be found in [17, pages 283-284] using the equivalently language of extensions of a group by a group. Here we present a different proof.

Theorem 2.10. Any finite group is isomorphic to an iteration of normalized crossed products of finite simple groups.

Any abelian finite group is isomorphic to an iteration of normalized twisted products of various \mathbb{Z}_{p_i} , where p_i are prime numbers.

Proof. Let E be a finite group of order n . We will prove by induction on n . If $n = 2$ then $E \cong \mathbb{Z}_2$ and we are done. Assume that $n > 2$. If E is simple there is nothing to prove. Assume that E has a proper normal subgroup $\{1\} \neq H \triangleleft E$. It follows from Theorem 2.6 that there exists a normalized crossed system $(H, E/H, \alpha, f)$ such that $E \cong H \#_{\alpha}^f E/H$. As H and E/H have order $< n$ we apply the induction. The Abelian case follows similarly if we apply Theorem 2.6 and the fact that an Abelian simple group is isomorphic to \mathbb{Z}_p , for a prime number p : in this case (14) shows that in the Abelian case the action α that arises is trivial, i.e. the crossed product between H and E/H is reduced to the twisted product $E \cong H \times^f E/H$. \square

The examples below show how the Theorem 2.10 is applied.

Examples 2.11. (1) Let $f : C_2 \times C_2 \rightarrow C_2$ given by $f(1, 1) = f(1, a) = f(a, 1) = 1$, $f(a, a) = a$. Then f is a 2-cocycle and $C_4 \cong C_2 \times^f C_2$, the twisted product of C_2 and C_2 .

(2) Let

$$\alpha : C_2 \rightarrow \text{Aut}(C_4), \quad \alpha(1) = \text{Id}_{C_4}, \quad \alpha(a)(x) = x^{-1}$$

for all $x \in C_4$ and

$$f : C_2 \times C_2 \rightarrow C_4, \quad f(1, 1) = f(1, a) = f(a, 1) = 1, \quad f(a, a) = b$$

where b is a generator of C_4 . Then (C_4, C_2, α, f) is a normalized crossed system and $C_4 \#_{\alpha}^f C_2 \cong Q$, the quaternion group Q of order 8. Thus the quaternion group Q can be presented as

$$Q \cong (C_2 \times^{f_1} C_2) \#_{\alpha}^{f_2} C_2$$

for a 2-cocycle f_1 , a weak action α and a α -cocycle f_2 .

Theorem 2.10 leads naturally to the question of associativity of crossed products.

Problem 2. Let H, G and K be three groups and (H, G, α, f) and $(H \#_{\alpha}^f G, K, \beta, g)$ two crossed systems. Under what circumstances do two crossed systems (G, K, β', g') and $(H, G \#_{\beta'}^{g'} K, \alpha', f')$ and an isomorphism of groups

$$(H \#_{\alpha}^f G) \#_{\beta}^g K \cong H \#_{\alpha'}^{f'} (G \#_{\beta'}^{g'} K) \text{ exist?}$$

2.3. Commutativity of Crossed Product

The results presented here are the counterpart at the level of groups of some theorems proved recently in [18] for crossed products of group actions on rings.

We shall fix $\Gamma = (H, G, \alpha, f)$, a normalized crossed system of groups. We define the group of invariants of the weak action α as follows

$$H^G := \{h \in H \mid g \triangleright h = h, \forall g \in G\}$$

Then H^G is a subgroup of H called the *subgroup of invariants* of the crossed system and hence we have the following extension of groups that is associated to a crossed system (H, G, α, f) :

$$H^G \leq H \cong H \times \{1\} \trianglelefteq H \#_{\alpha}^f G$$

Problem 3. Let (H, G, α, f) be a normalized crossed system. Give a necessary and sufficient condition for the category of representations of $H \#_{\alpha}^f G$ to be equivalently to the category of representations of H^G .

We compute now the center of a crossed product:

Proposition 2.12. Let (H, G, α, f) be a normalized crossed system. Then $(h, g) \in Z(H \#_{\alpha}^f G)$ if and only if the following three conditions hold

$$g \triangleright h' = h^{-1}h'h, \quad g \in Z(G) \tag{16}$$

$$(g' \triangleright h)f(g', g) = hf(g, g') \tag{17}$$

for any $h' \in H$ and $g' \in G$.

Proof. It follows from (10) that $(H \times \{1\}) \cup (\{1\} \times G)$ is a set of generators for $H \#_{\alpha}^f G$. Thus, $(h, g) \in Z(H \#_{\alpha}^f G)$ if and only if (h, g) commutes with $(h', 1)$ and $(1, g')$ for all $h' \in H$ and $g' \in G$. A direct computation shows that $(h, g) \cdot (1, g') = (1, g') \cdot (h, g)$ for all $g' \in G$ if and only if $g \in Z(G)$ and (17) holds. Similarly, using (7) we can show that $(h, g) \cdot (h', 1) = (h', 1) \cdot (h, g)$ for all $h' \in H$ if and only if the first condition in (16) holds. \square

The next corollary gives the center of a twisted product:

Corollary 2.13. Let H and G be two groups, $f : G \times G \rightarrow Z(H)$ a normalized 2-cocycle and $H \times^f G$ the twisted product of H and G . Then

$$Z(H \times^f G) = \{(h, g) \in Z(H) \times Z(G) \mid f(-, g) = f(g, -)\}$$

In particular, a twisted product $H \times^f G$ is an Abelian group if and only if H and G are Abelian groups and f is a symmetric 2-cocycle.

Using the subgroup H^G of invariants we give a description of the center of a crossed product having a symmetric cocycle:

Corollary 2.14. Let (H, G, α, f) be a normalized crossed system such that f is a symmetric α -cocycle. Then

$$Z(H \#_{\alpha}^f G) = \{(h, g) \in H^G \times Z(G) \mid g \triangleright h' = h^{-1}h'h, \forall h' \in H\}$$

The next result gives a necessary and sufficient condition for a crossed product to be an Abelian group.

Corollary 2.15. Let (H, G, α, f) be a normalized crossed system. Then $H \#_{\alpha}^f G$ is an Abelian group if and only if H and G are Abelian groups, α is the trivial action and f is a symmetric 2-cocycle.

Proof. Assume that $H\#_\alpha^f G$ is an Abelian group. Then H and G are Abelian groups. Using (16) of Proposition 2.12 we obtain that α is the trivial action and hence (17) shows that f is symmetric. The converse follows from Corollary 2.13. \square

Remark 2.16. Let (H, G, α, f) be a normalized crossed system. Then the centralizer of $H \cong H \times \{1\}$ in $H\#_\alpha^f G$ is given by

$$C_{H\#_\alpha^f G}(H) = \{(h, g) \mid g \triangleright x = h^{-1}xh, \forall x \in H\}.$$

If H is Abelian then α is a morphism of groups and $C_{H\#_\alpha^f G}(H) = H \times \text{Ker}(\alpha)$. Moreover, if G is also Abelian and f is a symmetric α -cocycle then $C_{H\#_\alpha^f G}(H)$ is an Abelian group.

3. CROSSED PRODUCT AS INITIAL AND FINAL OBJECT. APPLICATIONS

In this section $\Gamma = (H, G, \alpha, f)$ will be a normalized crossed system of groups. We shall prove that the crossed product $H\#_\alpha^f G$ is determined by a universal property in two ways: it can be viewed at the same time as an initial object in a certain category ${}_\Gamma\mathcal{C}$ and as a final object into another category \mathcal{C}_Γ . Define the category ${}_\Gamma\mathcal{C}$ as follows: the objects in ${}_\Gamma\mathcal{C}$ are pairs $(X, (u, v))$ where X is a group, $u : H \rightarrow X$ is a morphism of groups, $v : G \rightarrow X$ is a map such that the following two compatibility conditions hold:

$$v(g_1)v(g_2) = u(f(g_1, g_2))v(g_1g_2), \quad v(g)u(h) = u(g \triangleright h)v(g) \tag{18}$$

for all $g, g_1, g_2 \in G$ and $h \in H$. A morphism $f : (X, (u, v)) \rightarrow (X', (u', v'))$ in ${}_\Gamma\mathcal{C}$ is a morphism of groups $f : X \rightarrow X'$ such that $f \circ u = u'$ and $f \circ v = v'$. It can be checked that $(H\#_\alpha^f G, (i_H, i_G))$ is an object of ${}_\Gamma\mathcal{C}$, where i_H and i_G are the canonical inclusions $i_H : H \rightarrow H\#_\alpha^f G$, $i_H(h) = (h, 1)$ and $i_G : G \rightarrow H\#_\alpha^f G$, $i_G(g) = (1, g)$ for all $h \in H$ and $g \in G$.

Define the category \mathcal{C}_Γ as follows: the objects in \mathcal{C}_Γ are pairs $(X, (u, v))$ where X is a group, $v : X \rightarrow G$ is a morphism of groups, $u : X \rightarrow H$ is a map such that the following compatibility condition holds:

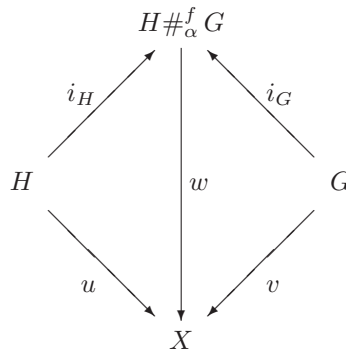
$$u(xy) = u(x)(v(x) \triangleright u(y))f(v(x), v(y)) \tag{19}$$

for all $x, y \in X$. A morphism $f : (X, (u, v)) \rightarrow (X', (u', v'))$ in \mathcal{C}_Γ is a morphism of groups $f : X \rightarrow X'$ such that $u' \circ f = u$ and $v' \circ f = v$. It can be checked that $(H\#_\alpha^f G, (\pi_H, \pi_G))$ is an object of \mathcal{C}_Γ where π_H and π_G are the canonical projections $\pi_H : H\#_\alpha^f G \rightarrow H$, $\pi_H(h, g) = h$ and $\pi_G : H\#_\alpha^f G \rightarrow G$, $\pi_G(h, g) = g$ for all $h \in H$ and $g \in G$.

Theorem 3.1. *Let $\Gamma = (H, G, \alpha, f)$ be a normalized crossed system of groups. Then:*

- (1) $(H\#_\alpha^f G, (i_H, i_G))$ is an initial object of ${}_\Gamma\mathcal{C}$.
- (2) $(H\#_\alpha^f G, (\pi_H, \pi_G))$ is a final object of \mathcal{C}_Γ .

Proof. (1) Let $(X, (u, v)) \in {}_\Gamma\mathcal{C}$. We have to prove that there exists a unique morphism of groups $w : H\#_\alpha^f G \rightarrow X$ such that the following diagram commutes:



Assume that w satisfies this condition. Then

$$\begin{aligned}
 w((h, g)) &= w((h, 1) \cdot (1, g)) = w((h, 1))w((1, g)) \\
 &= (w \circ i_H)(h)(w \circ i_G)(g) = u(h)v(g)
 \end{aligned}$$

for all $h \in H$ and $g \in G$ and this proves that w is unique. The existence of w can be proved as follows: define

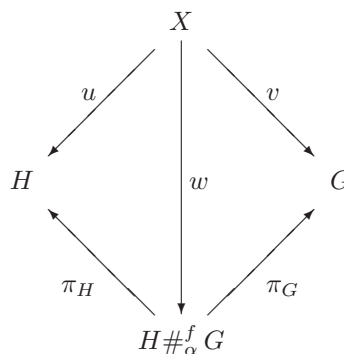
$$w : H \#_{\alpha}^f G \rightarrow X, \quad w(h, g) = u(h)v(g)$$

Then

$$\begin{aligned} w((h_1, g_1) \cdot (h_2, g_2)) &= w(h_1(g_1 \triangleright h_2)f(g_1, g_2), g_1g_2) \\ &= u(h_1)u(g_1 \triangleright h_2)u(f(g_1, g_2))v(g_1g_2) \\ &\stackrel{(18)}{=} u(h_1)u(g_1 \triangleright h_2)v(g_1)v(g_2) \\ &\stackrel{(18)}{=} u(h_1)v(g_1)u(h_2)v(g_2) = w(h_1, g_1)w(h_2, g_2) \end{aligned}$$

i.e. w is a morphism of groups. The fact that the diagram above commutes is left to the reader.

(2) Let $(X, (u, v)) \in \mathcal{C}_{\Gamma}$. We have to prove that there exists a unique morphism of groups $w : X \rightarrow H \#_{\alpha}^f G$ such that the following diagram commutes:



Assume that w satisfies this condition; then the commutativity of the diagrams gives that $w(x) = (u(x), v(x))$ for all $x \in X$, *i.e.* w is unique. For the existence of w we define $w : X \rightarrow H \#_{\alpha}^f G$, $w(x) = (u(x), v(x))$ for all $x \in X$. We have:

$$\begin{aligned} w(x)w(y) &= (u(x), v(x)) \cdot (u(y), v(y)) \\ &= \left(u(x)(v(x) \triangleright u(y))f(v(x), v(y)), v(x)v(y) \right) \\ &\stackrel{(19)}{=} (u(xy), v(xy)) = w(xy) \end{aligned}$$

i.e. w is a morphism of groups, as needed. □

The first application of Theorem 3.1 is the description of the (iso)morphism between a crossed product and a group.

Corollary 3.2. *Let X be a group and (H, G, α, f) be a normalized crossed system of groups. Then:*

- (1) *A map $w : H \#_{\alpha}^f G \rightarrow X$ is a morphism of groups if and only if there exists a pair (u, v) , where $u : H \rightarrow X$ is a morphism of groups, $v : G \rightarrow X$ is a map such that*

$$v(g_1)v(g_2) = u(f(g_1, g_2))v(g_1g_2), \tag{20}$$

$$v(g)u(h) = u(g \triangleright h)v(g), \tag{21}$$

$$w(h, g) = u(h)v(g) \tag{22}$$

for all $g, g_1, g_2 \in G$ and $h \in H$.

- (2) *A map $\psi : X \rightarrow H \#_{\alpha}^f G$ is a morphism of groups if and only if there exists a pair (u, v) , where $u : X \rightarrow H$ is a map, $v : X \rightarrow G$ a morphism of groups such that*

$$u(xy) = u(x)(v(x) \triangleright u(y))f(v(x), v(y)) \tag{23}$$

$$\psi(x) = (u(x), v(x)) \tag{24}$$

for all $x, y \in X$.

- (3) $w : H\#_{\alpha}^f G \rightarrow X$ given by (22) is an isomorphism of groups if and only if there exists a pair (r, s) , where $r : X \rightarrow G$ is a morphism of groups and a retraction of v (i.e. $r \circ v = Id_G$), $s : X \rightarrow H$ is a map that is a retraction of u (i.e. $s \circ u = Id_H$) such that:

$$\begin{aligned} s(xy) &= s(x)(r(x) \triangleright s(y))f(r(x), r(y)), & u(s(x))v(r(x)) &= x, \\ r(u(h)) &= 1, & s(v(g)) &= 1 \end{aligned}$$

for all $x \in X, h \in H$ and $g \in G$.

- (4) $\psi : X \rightarrow H\#_{\alpha}^f G$ given by (24) is an isomorphism of groups if and only if there exists a pair (r, s) , where $r : H \rightarrow X$ is a morphism of groups and a section of u (i.e. $u \circ r = Id_H$), $s : G \rightarrow X$ is a map that is a section of v (i.e. $v \circ s = Id_G$) such that:

$$\begin{aligned} s(g_1)s(g_2) &= r(f(g_1, g_2))s(g_1g_2), & s(g)r(h) &= r(g \triangleright h)s(g) \\ r(u(x))s(v(x)) &= x, & v(r(h)) &= 1, & u(s(g)) &= 1 \end{aligned}$$

for all $g_1, g_2, g \in G$ and $h \in H$.

Proof. (1) We shall use Theorem 3.1: if $w : H\#_{\alpha}^f G \rightarrow X$ is a morphism of groups then we take u and v given by $u(h) := w(h, 1)$ and $v(g) := w(1, g)$. Conversely, if u and v are given we define $w(h, g) := u(h)v(g)$.

(2) follows analogous from Theorem 3.1.

(3) $w : H\#_{\alpha}^f G \rightarrow X$ is an isomorphism of groups if and only if there exists $\psi : X \rightarrow H\#_{\alpha}^f G$ a morphism of groups such that $\psi \circ w = Id_{H\#_{\alpha}^f G}$ and $w \circ \psi = Id_X$. Using (2) ψ is a morphism of groups if and only if there exists $r : X \rightarrow G$ a morphism of groups and a map $s : X \rightarrow H$ such that:

$$s(xy) = s(x)(r(x) \triangleright s(y))f(r(x), r(y)), \quad \psi(x) = (s(x), r(x))$$

Now if we use $\psi \circ w = Id_{H\#_{\alpha}^f G}$ and $w \circ \psi = Id_X$ for the generators of $H\#_{\alpha}^f G$ the conclusion follows. Item (4) follows similarly. \square

In particular we can find the morphisms between two crossed products. If $\alpha' : G \rightarrow \text{Aut}(H)$ is another weak action we shall denote $\alpha(g)(h) = g \triangleright' h$.

Corollary 3.3. *Let (H, G, α, f) and (H, G, α', f') be two normalized crossed systems. There exists a bijection between the set of all morphisms of groups $\psi : H\#_{\alpha}^f G \rightarrow H\#_{\alpha'}^{f'} G$ and the set of all quadruples (u, r, v, s) , where $u : H \rightarrow H, r : G \rightarrow H, v : G \rightarrow G$ are three maps, and $s : H \rightarrow G$ is a morphism of groups such that*

$$\begin{aligned} v(g_1)v(g_2) &= s(f(g_1, g_2))v(g_1g_2) \\ v(g)s(h) &= s(g \triangleright h)v(g) \\ u(h_1h_2) &= u(h_1)(s(h_1) \triangleright' u(h_2))f'(s(h_1), s(h_2)) \end{aligned}$$

$$\begin{aligned} u(g \triangleright h) \left(s(g \triangleright h) \triangleright' r(g) \right) f' \left(s(g \triangleright h), v(g) \right) &= r(g)(v(g) \triangleright' u(h))f'(v(g), s(h)) \\ r(g_1)(v(g_1) \triangleright' r(g_2))f'(v(g_1), v(g_2)) &= u(f(g_1, g_2)) \left(s(f(g_1, g_2)) \triangleright' r(g_1g_2) \right) \\ & \quad f'(s(f(g_1, g_2)), v(g_1g_2)) \end{aligned}$$

Furthermore, the one to one correspondence is given such that $\psi : H\#_{\alpha}^f G \rightarrow H\#_{\alpha'}^{f'} G$ is given by the formula

$$\begin{aligned} \psi(h, g) &= (u(h), s(h)) \cdot (r(g), v(g)) \\ &= \left(u(h)(s(h) \triangleright' r(g))f'(s(h), v(g)), s(h)v(g) \right) \end{aligned}$$

and $u(1) = 1, v(1) = 1$ and $r(1) = 1$, for any such quadruple (u, r, v, s) .

Proof. Let $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$ be a morphism of groups. Using 1) of Corollary 3.2 for $X := H \#_{\alpha'}^{f'} G$ we get that there exists a unique pair (Φ_1, Φ_2) , where $\Phi_1 : H \rightarrow X$ is a morphism of groups and a map $\Phi_2 : G \rightarrow X$ such that

$$\psi(h, g) = \Phi_1(h)\Phi_2(g) \tag{25}$$

$$\Phi_2(g_1)\Phi_2(g_2) = \Phi_1(f(g_1, g_2))\Phi_2(g_1g_2) \tag{26}$$

$$\Phi_2(g)\Phi_1(h) = \Phi_1(g \triangleright h)\Phi_2(g) \tag{27}$$

for all $h \in H, g, g_1, g_2 \in G$. $\Phi_1 : H \rightarrow H \#_{\alpha'}^{f'} G$ is a morphism of groups: thus using 2) of Corollary 3.2 we get a unique pair (s, u) , where $s : H \rightarrow G$ is a morphism of groups, $u : H \rightarrow H$ is a map such that

$$\Phi_1(h) = (u(h), s(h)) \tag{28}$$

$$u(h_1h_2) = u(h_1)(s(h_1) \triangleright' u(h_2))f'(s(h_1), s(h_2)) \tag{29}$$

for all $h, h_1, h_2 \in H$. $\Phi_2 : G \rightarrow H \#_{\alpha'}^{f'} G$ is a map: hence there exists a unique pair of maps (r, v) , where $r := \pi_H \circ \Phi_2 : G \rightarrow H$ and $v := \pi_G \circ \Phi_2 : G \rightarrow G$ such that $\Phi_2(g) = (r(g), v(g))$, for all $g \in G$. To conclude, for any morphism of groups $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$ there exist unique quadruples (s, u, r, v) as above such that

$$\psi(h, g) = (u(h), s(h)) \cdot (r(g), v(g)) = \left(u(h)(s(h) \triangleright' r(g))f'(s(h), v(g)), s(h)v(g) \right)$$

Moreover, the compatibility conditions (25)–(29) are reduced to exactly the five compatibility conditions of the statement of the Corollary. Finally, if we specialize the first condition at $g_1 = g_2 = 1$, we obtain $v(1) = 1$. Then if we put $h_1 = h_2 = 1$ in the third relation we get that $u(1) = 1$ and if we let $g = 1$ and $h = 1$ in the fourth condition we obtain $r(1) = 1$. \square

The morphisms between two crossed products that stabilize the ends are much easier to describe:

Corollary 3.4. *Let (H, G, α, f) and (H, G, α', f') be two normalized crossed systems. There exists a bijection between the set of all (iso)morphisms of groups $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$ such that the diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & H & \xrightarrow{i_H} & H \#_{\alpha}^f G & \xrightarrow{\pi_G} & G & \longrightarrow & 1 \\ & & \downarrow Id_H & & \downarrow \psi & & \downarrow Id_G & & \\ 1 & \longrightarrow & H & \xrightarrow{i'_H} & H \#_{\alpha'}^{f'} G & \xrightarrow{\pi'_G} & G & \longrightarrow & 1 \end{array} \tag{30}$$

is commutative[†] and the set of all maps $r : G \rightarrow H$ such that

$$g \triangleright' h = r(g)^{-1}(g \triangleright h)r(g) \tag{31}$$

$$f'(g_1, g_2) = (g_1 \triangleright' r(g_2)^{-1})r(g_1)^{-1}f(g_1, g_2)r(g_1g_2) \tag{32}$$

for all $g, g_1, g_2 \in G$ and $h \in H$. Furthermore, the one to one correspondence is given such that $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$ is given by the formula

$$\psi(h, g) = (hr(g), g) \tag{33}$$

for all $h \in H$ and $g \in G$.

Proof. Let $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$ be a morphism of groups such that the above diagram is commutative. Thus there exist three maps $u : H \rightarrow H, r : G \rightarrow H, v : G \rightarrow G$ and a morphism of groups $s : H \rightarrow G$ such that the five relations of Corollary 3.3 hold and $\psi \circ i_H = i'_H \circ Id_H, Id_G \circ \pi_G = \pi'_G \circ \psi$. It follows that $u = Id_H, v = Id_G$ and $s(h) = 1$ for any $h \in H$. Hence, the five compatibility conditions of Corollary 3.3 are reduced to (31) and (32) and we are done. \square

[†]Such a morphism is necessarily an isomorphism of groups [13, Theorem 3.2.3].

Remark 3.5. Corollary 3.4 is a classification result: see Corollary 4.3 below for the exact statement. It generalizes the Schreier theorem [8, Theorem 7.34] that can be obtained if we let H to be an Abelian group and $\alpha' = \alpha$. In this case (31) holds and (32) can be written as:

$$f(g_1, g_2)f'(g_1, g_2)^{-1} = r(g_1)(g_1 \triangleright r(g_2))r(g_1, g_2)^{-1}$$

for all $g_1, g_2 \in G$ which means that $f'f^{-1}$ is coboundary.

Using Corollary 3.2 we can also describe the isomorphisms between two crossed products: the explicit description is left to the reader (we refer to [19] for full details). Corollary 3.3 can be used to compute all the (iso)morphism between all special cases of crossed products. We shall indicate only two relevant cases: first we shall compute the morphisms between a semidirect product and a twisted product and then we shall describe the morphisms between a crossed product and a direct product.

Corollary 3.6. *Let H and G be two groups, $f : G \times G \rightarrow Z(H)$ a normalized 2-cocycle and $\alpha : G \rightarrow \text{Aut}(H)$ a morphism of groups. There exists a bijection between the set of all morphisms of groups $\psi : H \rtimes_{\alpha} G \rightarrow H \rtimes^f G$ and the set of all quadruples (s, u, r, v) , where $s : H \rightarrow G$, $v : G \rightarrow G$ are morphisms of groups and $u : H \rightarrow H$, $r : G \rightarrow H$ are two maps such that:*

$$\begin{aligned} r(g_1g_2) &= r(g_1)r(g_2)f(v(g_1), v(g_2)) \\ u(h_1h_2) &= u(h_1)u(h_2)f(s(h_1), s(h_2)) \\ v(g)s(h) &= s(g \triangleright h)v(g) \\ r(g)u(h)f(v(g), s(h)) &= u(g \triangleright h)r(g)f(s(g \triangleright h), v(g)) \end{aligned}$$

for all $g, g_1, g_2 \in G$, $h, h_1, h_2 \in H$. Moreover, through the above bijection ψ is given by

$$\psi(h, g) = (u(h)r(g)f(s(h), v(g)), s(h)v(g))$$

for all $h \in H$ and $g \in G$.

Proof. We apply Corollary 3.3 in the case that f is a trivial cocycle and α' is a trivial action. □

Corollary 3.7. *Let (H, G, α, f) be a normalized crossed system. There exists a bijection between the set of all morphisms of groups $\psi : H \#_{\alpha}^f G \rightarrow H \times G$ and the set of all quadruples (s, u, r, v) , where $s : H \rightarrow G$, $u : H \rightarrow H$ are morphisms of groups, and $r : G \rightarrow H$, $v : G \rightarrow G$ are maps such that:*

$$\begin{aligned} v(g_1)v(g_2) &= s(f(g_1, g_2))v(g_1g_2), & r(g)u(h) &= u(g \triangleright h)r(g) \\ r(g_1)r(g_2) &= u(f(g_1, g_2))r(g_1g_2), & v(g)s(h) &= s(g \triangleright h)v(g) \end{aligned}$$

for all $g, g_1, g_2 \in G$, $h, h_1, h_2 \in H$. Moreover, through the above bijection ψ is given by

$$\psi(h, g) = (u(h)r(g), s(h)v(g))$$

for all $g \in G$, $h \in H$.

Proof. We apply Corollary 3.3 in the case that f' is a trivial cocycle and α' is a trivial action. □

The next Corollary describes the category of the representations of a crossed product $H \#_{\alpha}^f G$.

Corollary 3.8. *Let (H, G, α, f) be a normalized crossed system and k be a field. Then there exists an equivalence of categories between ${}_{k[H \#_{\alpha}^f G]} \mathcal{M}$ the category of left $k[H \#_{\alpha}^f G]$ -modules and the category of all triples (V, \bullet, \star) consisting of a left $k[H]$ -module (V, \bullet) and a k -linear map $\star : k[G] \otimes_k V \rightarrow V$ such that*

$$g_1 \star (g_2 \star v) = f(g_1, g_2) \bullet ((g_1g_2) \star v), \quad g \star (h \bullet v) = (g \triangleright h) \bullet (g \star v)$$

for all $g, g_1, g_2 \in G$, $h \in H$ and $v \in V$.

Proof. We apply (1) of Corollary 3.2 for $X = \text{Aut}_k(V)$, the group of automorphisms of a k -vector space V . □

The next Corollary is also of interest as it reminds us of the classical Clifford third problem of group representations:

Corollary 3.9. *Let X be a group, (H, G, α, f) a normalized crossed system and $u : H \rightarrow X$ a morphism of groups. Then there exists a morphism of groups $w : H \#_{\alpha}^f G \rightarrow X$ such that the following diagram*

$$\begin{array}{ccc} H & \xrightarrow{i_H} & H \#_{\alpha}^f G \\ \downarrow u & \searrow w & \\ X & & \end{array}$$

is commutative if and only if there exists a map $v : G \rightarrow X$ such that

$$v(g_1)v(g_2) = u(f(g_1, g_2))v(g_1g_2), \quad v(g)u(h) = u(g \triangleright h)v(g)$$

for all $g_1, g_2, g \in G$ and $h \in H$.

Proof. We apply (1) of Corollary 3.2. □

We shall consider now the exact sequence (11) and we can ask when $i_H : H \rightarrow H \#_{\alpha}^f G$ splits in the category of groups. For $X := H$ and $u := \text{Id}_H$ in the above corollary we obtain the answer:

Corollary 3.10. *Let (H, G, α, f) be a normalized crossed system. There exists a morphism of groups $w : H \#_{\alpha}^f G \rightarrow H$ such that $w \circ i_H = \text{Id}_H$ if and only if there exists a map $v : G \rightarrow H$ such that*

$$g \triangleright h = v(g)hv(g)^{-1}, \quad f(g_1, g_2) = v(g_1)v(g_2)v(g_1g_2)^{-1}$$

for all $g_1, g_2, g \in G$ and $h \in H$.

Dual to Corollary 3.9 we have:

Corollary 3.11. *Let (H, G, α, f) be a normalized crossed system, X be a group and $v : X \rightarrow G$ a morphism of groups. Then there exists $w : X \rightarrow H \#_{\alpha}^f G$ a morphism of groups such that the following diagram*

$$\begin{array}{ccc} & X & \\ & \swarrow w & \downarrow v \\ H \#_{\alpha}^f G & \xrightarrow{\pi_G} & G \longrightarrow 1 \end{array}$$

is commutative if and only if there exists a map $u : X \rightarrow H$ such that

$$u(xy) = u(x)(v(x) \triangleright u(y))f(v(x), v(y))$$

for all $x, y \in X$.

Proof. We apply (2) of Corollary 3.2. □

4. CATEGORICAL APPROACH: THE EXTENSION PROBLEM REVISED

Let H and G be two groups. We shall define three categories associated to H and G having the same class of objects, namely the set $\text{Crossed}(H, G)$ of all normalized crossed systems. We denote with $\mathcal{E}_1(H, G)$, $\mathcal{E}_2(H, G)$, $\mathcal{E}_3(H, G)$ the categories having as objects all normalized crossed systems (H, G, α, f) and morphisms defined as follows:

• A morphism $\psi : (\alpha, f) \rightarrow (\alpha', f')$ in $\mathcal{E}_1(H, G)$ is a morphism of groups $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$ such that the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & H & \xrightarrow{i_H} & H \#_{\alpha}^f G & \xrightarrow{\pi_G} & G \longrightarrow 1 \\ & & \downarrow Id_H & & \downarrow \psi & & \downarrow Id_G \\ 1 & \rightarrow & H & \xrightarrow{i'_H} & H \#_{\alpha'}^{f'} G & \xrightarrow{\pi'_G} & G \longrightarrow 1 \end{array}$$

is commutative.

• A morphism $(\eta, \psi, \gamma) : (\alpha, f) \rightarrow (\alpha', f')$ in $\mathcal{E}_2(H, G)$ is a triple (η, ψ, γ) , where $\eta : H \rightarrow H$, $\gamma : G \rightarrow G$ and $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$ are morphisms of groups such that the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & H & \xrightarrow{i_H} & H \#_{\alpha}^f G & \xrightarrow{\pi_G} & G \longrightarrow 1 \\ & & \downarrow \eta & & \downarrow \psi & & \downarrow \gamma \\ 1 & \rightarrow & H & \xrightarrow{i'_H} & H \#_{\alpha'}^{f'} G & \xrightarrow{\pi'_G} & G \longrightarrow 1 \end{array} \tag{34}$$

is commutative.

• A morphism $\psi : (\alpha, f) \rightarrow (\alpha', f')$ in $\mathcal{E}_3(H, G)$ is a morphism of groups $\psi : H \#_{\alpha}^f G \rightarrow H \#_{\alpha'}^{f'} G$.

Remarks 4.1. (1) The category $\mathcal{E}_1(H, G)$ is a groupoid. This category was used in the extension problem for groups. The category $\mathcal{E}_2(H, G)$ is a diagram-type category and, from a categorical point of view, defining the morphism as we have done in $\mathcal{E}_2(H, G)$ is more natural than the one in $\mathcal{E}_1(H, G)$. Finally, in order to classify all crossed product structures $H \#_{\alpha}^f G$ that can be constructed for two fixed groups H and G (*i.e.* in order to solve the classification part of Problem 1 we have to deal with the morphisms as we defined them in $\mathcal{E}_3(H, G)$).

(2) $\mathcal{E}_1(H, G)$ is a subcategory of $\mathcal{E}_3(H, G)$ and there exist functors connecting the above categories as follows:

$$\begin{aligned} F_{23} : \mathcal{E}_2(H, G) &\rightarrow \mathcal{E}_3(H, G), & F_{23}((\alpha, f)) &:= (\alpha, f), & F_{23}(\eta, \psi, \gamma) &:= \psi \\ F_{12} : \mathcal{E}_1(H, G) &\rightarrow \mathcal{E}_2(H, G), & F_{12}(\alpha, f) &:= (\alpha, f), & F_{12}(\psi) &:= (Id_H, \psi, Id_G) \end{aligned}$$

Having in mind that the forgetful type functors usually have adjoints we can ask:

Problem 4. Let H and G be groups. Do the above functors F_{12} , F_{23} or the inclusion functor $i : \mathcal{E}_1(H, G) \rightarrow \mathcal{E}_3(H, G)$ have right or left adjoints?

The classification part of the extending problem can be restated as follows: describe the skeleton of the category $\mathcal{E}_1(H, G)$. The following is of course natural and more general:

Let H and G be groups. Describe the skeleton of the categories $\mathcal{E}_i(H, G)$, $i = 1, 2, 3$.

The skeleton of the category $\mathcal{E}_1(H, G)$ is obtained from Corollary 3.4 as follows [†] :

Definition 4.2. Two normalized crossed systems (H, G, α, f) , (H, G, α', f') are called 1 - equivalently and we denote it by $(H, G, \alpha, f) \approx_1 (H, G, \alpha', f')$ if there exists a map $r : G \rightarrow H$ such that

$$\begin{aligned} g \triangleright' h &= r(g)^{-1}(g \triangleright h)r(g) \\ f'(g_1, g_2) &= (g_1 \triangleright' r(g_2)^{-1})r(g_1)^{-1}f(g_1, g_2)r(g_1g_2) \end{aligned}$$

for all $g, g_1, g_2 \in G$ and $h \in H$.

Corollary 3.4 shows that $(H, G, \alpha, f) \approx_1 (H, G, \alpha', f')$ if and only if there exists $\psi : (\alpha, f) \rightarrow (\alpha', f')$ an isomorphism in $\mathcal{E}_1(H, G)$. Thus \approx_1 is an equivalence relation on the set of all normalized crossed systems Crossed (H, G) and we have proved that:

[†]In [9] it is stated that the skeleton of the category $\mathcal{E}_1(H, G)$ is classified by weak 2-functors $G \rightarrow AUT(H)$.

Corollary 4.3. *Let H and G be two groups. There exists a bijection between the set of objects of the skeleton of the category $\mathcal{E}_1(H, G)$ and the quotient set $\text{Crossed}(H, G)/\approx_1$.*

Now we shall describe the skeleton of the category $\mathcal{E}_2(H, G)$. First we need the following:

Proposition 4.4. *Let H, G be two groups. Then $(\eta, \psi, \gamma) : (\alpha, f) \rightarrow (\alpha', f')$ is a morphism of $\mathcal{E}_2(H, G)$ if and only if $\eta : H \rightarrow H, \gamma : G \rightarrow G$ are morphisms of groups and there exists a unique map $r : G \rightarrow H$ such that:*

$$\eta(g \triangleright h)r(g) = r(g)(\gamma(g) \triangleright' \eta(h)) \tag{35}$$

$$r(g_1)(\gamma(g_1) \triangleright' r(g_2))f'(\gamma(g_1), \gamma(g_2)) = \eta(f(g_1, g_2))r(g_1g_2) \tag{36}$$

$$\psi(h, g) = (\eta(h)r(g), \gamma(g)) \tag{37}$$

for all $h \in H, g, g_1, g_2 \in G$.

Proof. Since $\psi : H\#_{\alpha}^f G \rightarrow H\#_{\alpha'}^{f'} G$ is a morphism of groups, from Corollary 3.3 there exists a unique quadruple (u, r, v, s) , where $u : H \rightarrow H, r : G \rightarrow H, v : G \rightarrow G$ are three maps, and $s : H \rightarrow G$ is a morphism of groups satisfying the compatibility conditions from Corollary 3.3. On the other hand, the diagram (34) is commutative: thus $\psi(h, 1) = (\eta(h), 1) = (u(h), s(h))$ and $\gamma(g) = s(h)v(g)$. It follows that $u = \eta, v = \gamma$ and $s(h) = 1$ for any $h \in H$. Now, the first three compatibility conditions of Corollary 3.3 are equivalent to γ and η being morphisms of groups (as u and v are morphisms and $s(h) = 1$ for all h), and the last two compatibility conditions of Corollary 3.3 are reduced to the conditions (35)–(36). \square

Corollary 4.5. *Let H and G be two groups. Then $(\eta, \psi, \gamma) : (\alpha, f) \rightarrow (\alpha', f')$ is an isomorphism in $\mathcal{E}_2(H, G)$ if and only if $\eta : H \rightarrow H, \gamma : G \rightarrow G$ are isomorphisms of groups and there exists a unique map $t : G \rightarrow H$ such that:*

$$g \triangleright' h = \eta(t(g)(\gamma^{-1}(g) \triangleright \eta^{-1}(h))t(g)^{-1}) \tag{38}$$

$$f'(g_1, g_2) = \eta(t(g_1)(\gamma^{-1}(g_1) \triangleright t(g_2))f(\gamma^{-1}(g_1), \gamma^{-1}(g_2))t(g_1g_2)^{-1}) \tag{39}$$

for all $h \in H, g, g_1, g_2 \in G$. Moreover, the isomorphism $\psi : H\#_{\alpha}^f G \rightarrow H\#_{\alpha'}^{f'} G$ is given by the formula:

$$\psi(h, g) = (\eta(ht(\gamma(g))^{-1}), \gamma(g))$$

for all $h \in H$ and $g \in G$.

Proof. $\psi : H\#_{\alpha}^f G \rightarrow H\#_{\alpha'}^{f'} G$ given by (37) is an isomorphism if and only if it is bijective. A morphism of groups $\psi^{-1} : H\#_{\alpha'}^{f'} G \rightarrow H\#_{\alpha}^f G$ that makes the diagram (34) commutative has the form

$$\psi^{-1}(h, g) = (\eta^{-1}(h)t(g), \gamma^{-1}(g))$$

for a unique map $t : G \rightarrow H$ that satisfies two compatibility conditions similar to (35) and (36). Now, if we write $\psi \circ \psi^{-1} = \text{Id}_{H\#_{\alpha'}^{f'} G}$ and $\psi^{-1} \circ \psi = \text{Id}_{H\#_{\alpha}^f G}$ on the set of generators $(h, 1)$ and $(1, g)$ we obtain the relations

$$\eta^{-1}(r(g))t(\gamma(g)) = 1, \quad \eta(t(g))r(\gamma^{-1}(g)) = 1$$

or equivalently

$$r(g) = \eta(t(\gamma(g))^{-1})$$

for all $g \in G$. With this r the compatibility conditions (35)–(36) give exactly (38)–(39). \square

Remark 4.6. In particular, if we specialize Corollary 4.5 for (H, G, α, f) , the trivial crossed system, we obtain a necessary and sufficient condition for a crossed product to be isomorphic to a direct product in the category $\mathcal{E}_2(H, G)$. More precisely, $\psi : H \times G \rightarrow H\#_{\alpha}^f G$ is an isomorphism in $\mathcal{E}_2(H, G)$ if and only if there exists a pair (η, t) , where $\eta \in \text{Aut}(H)$ is an automorphism of $H, t : G \rightarrow H$ is a map such that:

$$g \triangleright' h = \eta(t(g)\eta^{-1}(h)t(g)^{-1})$$

$$f'(g_1, g_2) = \eta(t(g_1)t(g_2)t(g_1g_2)^{-1})$$

for all $h \in H, g, g_1, g_2 \in G$.

Definition 4.7. Two normalized crossed systems (H, G, α, f) , (H, G, α', f') are called 2 - equivalently and we denote it by $(H, G, \alpha, f) \approx_2 (H, G, \alpha', f')$ if there exists a triple (η, γ, t) , where $\eta : H \rightarrow H$, $\gamma : G \rightarrow G$ are isomorphisms of groups, $t : G \rightarrow H$ is a map such that the compatibility conditions (38) and (39) hold.

Corollary 4.5 shows that $(H, G, \alpha, f) \approx_2 (H, G, \alpha', f')$ if and only if there exists $(\eta, \psi, \gamma) : (\alpha, f) \rightarrow (\alpha', f')$ an isomorphism in $\mathcal{E}_2(H, G)$. Thus \approx_2 is an equivalence relation on the set of all crossed systems $\text{Crossed}(H, G)$ and we have proved the following classification result that is a general Schreier's type theorem:

Theorem 4.8. *Let H and G be two groups. There exists a bijection between the set of objects of the skeleton of the category $\mathcal{E}_2(H, G)$ and the quotient set $\text{Crossed}(H, G)/\approx_2$.*

We end the paper with the following:

Problem 5. *Construct the crossed product for groupoids and generalize the results presented in this paper to the level of groupoids.*

We recall that a groupoid is a small category in which any morphism is an isomorphism. This way, groups are groupoids with only one object. The construction of the crossed product for groupoids must be made in such a way that its generalization to the level of Hopf algebroids agrees with the one recently constructed in [10].

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